

SPECTRAL MULTIPLICITY FOR MAASS NEWFORMS OF NON-SQUAREFREE LEVEL

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ABSTRACT. We show that if a positive integer q has $s(q)$ odd prime divisors p for which p^2 divides q , then a positive proportion of the Laplacian eigenvalues of Maaß newforms of weight 0, level q , and principal character occur with multiplicity at least $2^{s(q)}$. Consequently, the new part of the discrete spectrum of the Laplacian on $\Gamma_0(q)\backslash\mathbb{H}$ cannot be simple for any odd non-squarefree integer q . This generalises work of Strömberg, who proved this for $q = 9$ by different methods.

1. INTRODUCTION

1.1. Strömberg's Results for Maaß Forms on $\Gamma_0(9)\backslash\mathbb{H}$. Let q be a positive integer, let ε be a Dirichlet character modulo q , and let $\mathcal{M}(q, \varepsilon)$ denote the vector space of Maaß cusp forms of weight 0, level q , and character ε . For $\lambda \geq 0$, we let $\mathcal{M}(q, \varepsilon, \lambda)$ denote the subspace of $\mathcal{M}(q, \varepsilon)$ consisting of Maaß cusp forms with Laplacian eigenvalue $\lambda = 1/4 + t^2$, where the spectral parameter it is in $i\mathbb{R} \cup (-1/2, 1/2)$. In each case, we let $\mathcal{M}^{\text{new}}(q, \varepsilon)$ and $\mathcal{M}^{\text{new}}(q, \varepsilon, \lambda)$ denote the subspaces spanned by newforms. When $\varepsilon = \varepsilon_{0(q)}$ is the principal character modulo q , we simply write $\mathcal{M}^{\text{new}}(\Gamma_0(q))$ and $\mathcal{M}^{\text{new}}(\Gamma_0(q), \lambda)$, where $\Gamma_0(q)$ is the congruence subgroup of $\text{SL}_2(\mathbb{Z})$ consisting of matrices whose lower left entry is divisible by q .

Given a primitive Dirichlet character ε' modulo q' and a Maaß newform $\varphi \in \mathcal{M}(q, \varepsilon)$ with Hecke eigenvalues $\lambda_\varphi(p)$ at each prime p , the twist of φ by ε' is the *newform* $\varphi \otimes \varepsilon'$ with Hecke eigenvalues equal to $\varepsilon(p)\lambda_\varphi(p)$ for every prime p not dividing $q'q$; this Maaß newform is of weight 0, level dividing $q'q$, and character induced by the same primitive character as $\varepsilon'^2\varepsilon$. Crucially, twisting by a character leaves the Laplacian eigenvalue of a Maaß form unchanged.

In [Str12], Strömberg proves some striking results about Maaß newforms on $\Gamma_0(9)\backslash\mathbb{H}$ of principal character $\varepsilon_{0(9)}$. He gives an orthogonal decomposition of the space $\mathcal{M}^{\text{new}}(\Gamma_0(9), \lambda)$ with respect to the Petersson inner product on $\Gamma_0(9)\backslash\mathbb{H}$:

$$\begin{aligned} & \mathcal{M}^{\text{new}}(\Gamma_0(9), \lambda) \\ &= (\mathcal{M}^{\text{new}}(\text{SL}_2(\mathbb{Z}), \lambda) \otimes \varepsilon_{\text{quad}(3)}) \oplus (\mathcal{M}^{\text{new}}(\Gamma_0(3), \lambda) \otimes \varepsilon_{\text{quad}(3)}) \oplus \mathcal{M}^{\text{new}}(\Gamma^3, \lambda)|_{V_3}. \end{aligned}$$

Here $\varepsilon_{\text{quad}(3)}$ denotes the unique quadratic character modulo 3 and $\mathcal{M}^{\text{new}}(\Gamma^3, \lambda)$ denotes the new space of Maaß cusp forms on $\Gamma^3\backslash\mathbb{H}$ having trivial congruence character and Laplacian eigenvalue λ . The congruence subgroup Γ^3 is

$$\Gamma^3 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) : ab + cd \equiv 0 \pmod{3} \right\}.$$

This is of index 3 in $\text{PSL}_2(\mathbb{Z})$ and contains the principal congruence subgroup $\Gamma(3)$, and as $\Gamma(3)$ is conjugate to $\Gamma_0(9)$ by $V_3 := \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \in \text{PGL}_2(\mathbb{Q})$, Maaß forms on $\Gamma^3\backslash\mathbb{H}$ therefore lift to Maaß forms on $\Gamma_0(9)\backslash\mathbb{H}$. More precisely, given a Maaß cusp form $\psi(z) \in \mathcal{M}(\Gamma^3, \lambda)$, the function $\psi|_{V_3}(z) := \psi(3z)$ is a Maaß cusp form in

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$\mathcal{M}(\Gamma_0(9), \lambda)$. The new space of $\mathcal{M}(\Gamma^3)$ consists of elements orthogonal to direct embeddings in $\mathcal{M}(\Gamma^3)$ of cusp forms in $\mathcal{M}(\mathrm{SL}_2(\mathbb{Z}))$.

In particular, Strömberg concludes that no newform in $\mathcal{M}^{\mathrm{new}}(\Gamma_0(9))$ is “genuinely new” in the sense that it does not arise as a twist of a newform of lower level or as the embedding of a newform on $\Gamma' \backslash \mathbb{H}$ for some congruence subgroup Γ' of smaller index in $\mathrm{PSL}_2(\mathbb{Z})$. Furthermore, letting $\lambda = \lambda_\varphi$ denote the Laplacian eigenvalue of a Maaß cusp form φ , Strömberg uses the Selberg trace formula to prove a Weyl law of the form

$$(1) \quad \frac{\#\{\psi \in \mathcal{M}^{\mathrm{new}}(\Gamma^3) : \lambda_\psi \leq T\}}{\#\{\varphi \in \mathcal{M}^{\mathrm{new}}(\Gamma_0(9)) : \lambda_\varphi \leq T\}} = \frac{2}{5} + o(1) \quad \text{as } T \rightarrow \infty,$$

so that two-fifths of the newforms in $\mathcal{M}^{\mathrm{new}}(\Gamma_0(9))$ arise from members of $\mathcal{M}^{\mathrm{new}}(\Gamma^3)$. Here of course when we write $\psi \in \mathcal{M}^{\mathrm{new}}(\Gamma^3)$ or $\varphi \in \mathcal{M}^{\mathrm{new}}(\Gamma_0(9))$, we mean that ψ or φ is a newform, and in particular an element of the orthogonal basis of these spaces.

Strömberg then shows that the new space of Maaß cusp forms on $\Gamma^3 \backslash \mathbb{H}$ of trivial congruence character further decomposes into two orthogonal parts according to whether the eigenvalue of $\psi \in \mathcal{M}^{\mathrm{new}}(\Gamma^3, \lambda)$ with respect to the action of the matrix $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $e^{2\pi i/3}$ or $e^{-2\pi i/3}$. Moreover, one can map each of these spaces to the other via the action of $J := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{R})$ (where the action of a matrix with negative determinant means that we replace z by \bar{z}), and this mapping preserves the Laplacian eigenvalue. So for every newform $\psi \in \mathcal{M}^{\mathrm{new}}(\Gamma^3)$, there exists a distinct newform $\psi' \in \mathcal{M}^{\mathrm{new}}(\Gamma^3)$ for which $\lambda_\psi = \lambda_{\psi'}$. By (1), Strömberg concludes the following.

Proposition 1.1 (Strömberg [Str12, Proposition 1.3]). *At least two-fifths of the newforms in $\mathcal{M}^{\mathrm{new}}(\Gamma_0(9))$ have repeated Laplacian eigenvalues.*

This result is of particular interest is due to its relation to a folklore conjecture on the simplicity of the spectrum of the Laplacian on $\Gamma_0(q) \backslash \mathbb{H}$. The discrete spectrum is precisely the set of eigenvalues of the Laplacian of Maaß cusp forms on $\Gamma_0(q)$ of principal character $\varepsilon_{0(q)}$. It is known that the multiplicity of an eigenvalue λ of the Laplacian on $\Gamma_0(q) \backslash \mathbb{H}$ is at most $C_q \sqrt{\lambda} / \log \lambda$ for some constant C_q dependent only on q ; cf. [Sar03, §4]. When $q = 1$, Cartier [Car71] conjectured that the discrete spectrum is simple, based on limited numerical evidence; more numerical calculations were performed by Steil [Ste94] that gave further support to this conjecture. For $q > 1$, the discrete spectrum is not simple due to newforms and oldforms having the same eigenvalue. Nevertheless, one may well ask whether this is the only cause of spectral multiplicity, so that the so-called new part of the discrete spectrum consisting of eigenvalues of the Laplacian associated to newforms is simple.

Question. *Is the new part of the discrete spectrum of the Laplacian on $\Gamma_0(q) \backslash \mathbb{H}$ simple?*

For squarefree q , this is certainly expected to be true; see, for example, the work of Bolte and Johansson [BJ99a], [BJ99b] and Strömbergsson [Str01] on the spectral correspondence between the new part of the discrete spectrum of the Laplacian on $\Gamma_0(q) \backslash \mathbb{H}$ and eigenvalues of the automorphic Laplacian for the group of units of norm one in a maximal order in an indefinite quaternion division algebra over \mathbb{Q} . On the other hand, it is known by purely representation-theoretic means that the discrete spectrum of the Laplacian on $\Gamma(p) \backslash \mathbb{H}$, where $\Gamma(p)$ is the principal congruence subgroup modulo an odd prime p , contains infinitely many eigenvalues with multiplicity at least $\frac{1}{2}(p + (-1)^{(p-1)/2})$ [Ran81].

Strömberg's result gives a value $q = 9$ for which the new part of the discrete spectrum of the Laplacian on $\Gamma_0(q)\backslash\mathbb{H}$ has many eigenvalues of multiplicity greater than one, and a lower bound for the number of such values. His proof passes through the subgroup Γ^3 by decomposing $\mathcal{M}^{\text{new}}(\Gamma_0(9))$ into orthogonal subspaces of twists of newforms of lower level and embeddings of newforms coming from congruence subgroups of smaller index in $\text{PSL}_2(\mathbb{Z})$, then looking at the normalisers of these intermediate congruence subgroups and searching for operators that decompose these spaces into further orthogonal subspaces that can be mapped to each other while preserving the eigenvalue of the Laplacian.

One may well ask whether this generalises to other positive integers q that are not squarefree. Strömberg [Str12, Section 8] studies the cases $p = 2, 5, 7$ of $\Gamma_0(p^2)$ additionally, albeit in some brevity. For $p = 2$, he states that

$$\mathcal{M}^{\text{new}}(\Gamma_0(4), \lambda) = \mathcal{M}^{\text{new}}(\Gamma^2, \lambda),$$

where Γ^2 is a certain subgroup of $\text{PSL}_2(\mathbb{Z})$ of index 2 that contains $\Gamma(2)$. A similar result seems to hold for $p = 5$, where one must now include twists of newforms in $\mathcal{M}^{\text{new}}(5, \varepsilon_{\text{quad}(5)}, \lambda)$ by each of the two Dirichlet characters modulo 5 of order 4. Again, there is a single intermediate subgroup Γ^5 between $\Gamma(5)$ and $\text{PSL}_2(\mathbb{Z})$ to consider, and newforms arising from this subgroup seem to occur with multiplicity 2, though Strömberg does not determine the operator on $\mathcal{M}^{\text{new}}(\Gamma^5, \lambda)$ that yields this multiplicity. Finally, for $p = 7$ the situation becomes more complicated, as there are several intermediate subgroups Γ' between $\Gamma(7)$ and $\text{PSL}_2(\mathbb{Z})$ that give rise to newforms that lift to newforms on $\Gamma_0(49)$. Furthermore, Strömberg numerically encounters Maaß newforms on $\Gamma_0(49)\backslash\mathbb{H}$ that do not seem to arise as embeddings of newforms from $\mathcal{M}^{\text{new}}(\Gamma', \lambda)$ into $\mathcal{M}^{\text{new}}(\Gamma_0(49), \lambda)$, though he remarks that he has not accounted for the possibility that such as-yet unclassified newforms do not arise from $\mathcal{M}^{\text{new}}(\Gamma', \varepsilon, \lambda)$ with ε a congruence character of Γ' .

1.2. Main Results. On closer inspection, it becomes clear that Strömberg's work actually deals with three separate questions.

Question 1. *Can one show whether a twist-minimal newform of principal character arises as a newform on $\Gamma'\backslash\mathbb{H}$ for some intermediate congruence subgroup Γ' of smaller index?*

Here twist-minimal means that a newform cannot be twisted by a Dirichlet character to yield a newform of lower level.

Question 2. *Can one count the number of Maaß newforms of level q of principal character and Laplacian eigenvalue λ that can be twisted by some Dirichlet character to give another newform of level q and principal character?*

Note that by the definition of a twist, such a character must be quadratic.

Question 3. *Can one give a lower bound in terms of q for the largest possible multiplicity of an eigenvalue in the discrete spectrum of the Laplacian on $\Gamma_0(q)\backslash\mathbb{H}$?*

While Strömberg answers Questions 2 and 3 for $q = 9$ by first dealing with Question 1, we find that the latter two questions can be addressed independently of the former question. Indeed, in [Str12, Section 6], Strömberg shows that the two orthogonal parts of $\mathcal{M}^{\text{new}}(\Gamma^3, \lambda)$, after embedding in $\mathcal{M}^{\text{new}}(\Gamma_0(9), \lambda)$, are related by twisting by the primitive quadratic character $\varepsilon_{\text{quad}(3)}$ modulo 3. So Question 2 is, in actuality, asking for a description of $\mathcal{M}^{\text{new}}(\Gamma_0(q), \lambda) \otimes \varepsilon_{\text{quad}(q')}$ for each q' dividing q for which there exists a primitive quadratic Dirichlet character $\varepsilon_{\text{quad}(q')}$, a problem first studied by Atkin and Lehner [AL70] and Atkin and Li [AL78] in the setting of holomorphic newforms, and this requires no knowledge of the former

question. For Question 1, on the other hand, there seems to be little in the literature dealing with this problem; it appears to be a more challenging issue that reduces to understanding restrictions of irreducible representations of $\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$ to certain subgroups.

We prove the following result that characterises Maaß newforms of principal character that are level-invariant under twisting by a quadratic character.

Theorem 1.2. *Let q be a positive integer, and let p be an odd prime dividing q . Let $\varepsilon_{\mathrm{quad}(p)}$ denote the unique quadratic character modulo p . Then if $p \parallel q$,*

$$\mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda) \cap (\mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda) \otimes \varepsilon_{\mathrm{quad}(p)}) = \{0\}.$$

If $p^2 \parallel q$,

$$\begin{aligned} \mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda) \cap (\mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda) \otimes \varepsilon_{\mathrm{quad}(p)}) \\ = \mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda) \ominus (\mathcal{M}^{\mathrm{new}}(\Gamma_0(qp^{-1}), \lambda) \otimes \varepsilon_{\mathrm{quad}(p)}) \\ \ominus (\mathcal{M}^{\mathrm{new}}(\Gamma_0(qp^{-2}), \lambda) \otimes \varepsilon_{\mathrm{quad}(p)}). \end{aligned}$$

If $p^3 \mid q$,

$$\mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda) \cap (\mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda) \otimes \varepsilon_{\mathrm{quad}(p)}) = \mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda).$$

We prove Theorem 1.2 in Section 3, and then generalise this result to allow twists by the unique primitive quadratic character $\varepsilon_{\mathrm{quad}(q')}$ modulo q' for any odd squarefree divisor q' of q . We find that these issues are better addressed in the adèlic setting, for then one can resolve this problem locally by studying generic irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Moreover, we can generalise Theorem 1.2 to Hilbert modular forms at one fell swoop, and indeed to modular forms over any number field. By the well-known correspondence between newforms and automorphic representations (see, for example, [GH11, Theorem 13.8.9]), it suffices to characterise the cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ whose archimedean component is a principal series representation such that π is conductor-invariant under twisting by a quadratic character of $\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}^{\times}$. As such, we cover in Section 2 the relationship between automorphic representations and representations of local fields and follow this with a classification of the behaviour of such local representations under twisting by quadratic characters.

Via the Weyl law for $\Gamma_0(q) \backslash \mathbb{H}$, Theorem 1.2 shows that a positive proportion of Maaß newforms of non-squarefree level q and principal character are level-invariant under twisting by a quadratic character. This, however, does not force spectral multiplicity for $\Gamma_0(q) \backslash \mathbb{H}$: we must eliminate the possibility that $\varphi \otimes \varepsilon_{\mathrm{quad}(p)} = \varphi$ for every newform $\varphi \in \mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda) \cap (\mathcal{M}^{\mathrm{new}}(\Gamma_0(q), \lambda) \otimes \varepsilon_{\mathrm{quad}(p)})$. Newforms that satisfy $\varphi \otimes \varepsilon = \varphi$ for some Dirichlet character ε are said to be monomial or of CM-type. In Section 4, we give an upper bound for the number of monomial newforms on $\Gamma_0(q) \backslash \mathbb{H}$ with Laplacian eigenvalue λ at most T with T tending to infinity.

Section 5 deals with the Weyl law for Maaß newforms. We show that the monomial newforms have density zero in the set of all newforms of level q and principal character. This allows us to count the number of Maaß newforms of level q and principal character whose twist by a quadratic character $\varepsilon_{\mathrm{quad}(q')}$ is a different Maaß newform of the same level and character; we denote by $\mathcal{M}^{\mathrm{new}}(\Gamma_0(q))_{\mathrm{nonmon}(\varepsilon_{\mathrm{quad}(q')})}$ the vector subspace of $\mathcal{M}^{\mathrm{new}}(\Gamma_0(q))$ spanned by such newforms.

Theorem 1.3. *Let q and q' be positive integers with q' odd and squarefree. Let $\varepsilon_{\mathrm{quad}(q')}$ denote the unique primitive quadratic character modulo q' . Then if there*

exists a prime p dividing q' such that p^2 does not divide q ,

$$\mathcal{M}^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q')})} = \{0\},$$

whereas if p^2 divides q for every prime p dividing q' , we have that

$$\frac{\#\left\{\varphi \in \mathcal{M}^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q')})} : \lambda_\varphi \leq T\right\}}{\#\left\{\varphi \in \mathcal{M}^{\text{new}}(\Gamma_0(q)) : \lambda_\varphi \leq T\right\}} = \prod_{\substack{p|q' \\ p^2 \nmid q}} \left(1 - \frac{p}{p^2 - p - 1}\right) + o_q(1)$$

as T tends to infinity, where the error term depends only on q , and each φ is a newform, and in particular an element of the orthogonal basis of $\mathcal{M}^{\text{new}}(\Gamma_0(q))$.

Moreover, the same holds if we replace $\mathcal{M}^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q')})}$ by

$$\bigcap_{\substack{q^*|q' \\ q^* > 1}} \mathcal{M}^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q^*)})}.$$

As φ and $\varphi \otimes \varepsilon_{\text{quad}(q')}$ have the same Laplacian eigenvalue λ , this means that every Laplacian eigenvalue corresponding to a newform in $\mathcal{M}^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q')})}$ has multiplicity at least 2. When $q = 9$ and $q' = 3$, this shows that two-fifths of the new part of the discrete spectrum of the Laplacian on $\Gamma_0(9) \backslash \mathbb{H}$ have multiplicity at least 2, thereby reproving a result of Strömberg [Str12, Proposition 1.3]. When $q = p^3$ is the cube of an odd prime, then Theorem 1.3 implies that eigenvalues with multiplicity at least 2 constitute almost all of the new part of the discrete spectrum of the Laplacian on $\Gamma_0(q) \backslash \mathbb{H}$, with the only eigenvalues having multiplicity 1 corresponding to monomial newforms.

More generally, the space

$$\bigcap_{\substack{q^*|q' \\ q^* > 1}} \mathcal{M}^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q^*)})}$$

remains unchanged under twisting by each unique primitive quadratic character $\varepsilon_{\text{quad}(q^*)}$ modulo q^* with $q^* | q'$ and $q^* > 1$. As this space does not contain any monomial newforms, we deduce the following corollary.

Corollary 1.4. *Let q be a positive integer, and let $s(q)$ denote the number of distinct odd primes p for which p^2 divides q . Then a set of eigenvalues of the new part of the discrete spectrum of the Laplacian on $\Gamma_0(q) \backslash \mathbb{H}$ with density*

$$\prod_{p^2 \nmid q} \left(1 - \frac{p}{p^2 - p - 1}\right)$$

has multiplicity at least $2^{s(q)}$.

Proof. Set q' to be the largest odd squarefree integer dividing q such that every prime p dividing q' is such that p^2 divides q . Then every newform φ in

$$\bigcap_{\substack{q^*|q' \\ q^* > 1}} \mathcal{M}^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q^*)})}$$

has the same Laplacian eigenvalue as $\varphi \otimes \varepsilon_{\text{quad}(q^*)}$ for every $q^* | q'$ with $q^* > 1$. This gives $2^{s(q)}$ different newforms with the same Laplacian eigenvalue. \square

In particular, we see that the new part of the discrete spectrum of the Laplacian on $\Gamma_0(q) \backslash \mathbb{H}$ will contain eigenvalues with high multiplicity if q is highly composite and non-squarefree.

2. SOME BACKGROUND ON AUTOMORPHIC REPRESENTATIONS

2.1. Local and Global Definitions. We recall some background information on automorphic representations of GL_2 ; see [Bum97], [Gel75], [GH11], and [JL70] for further details.

Let F be an algebraic number field and let \mathbb{A}_F denote the ring of adèles of F . A place of F will be denoted by v , and F_v will denote the corresponding local field. We let S_∞ and S_f denote the set of archimedean and nonarchimedean places of F respectively; we recall that a nonarchimedean place v corresponds to a prime ideal \mathfrak{p} of F via the identification $\mathfrak{p} = \mathfrak{p}_v \cap \mathcal{O}_F$, where \mathfrak{p}_v is the maximal ideal of the ring of integers \mathcal{O}_v of the local field F_v .

For a nonarchimedean place v of F , let $\omega_v: F_v^\times \rightarrow \mathbb{C}^\times$ be a character of F_v^\times . We have that $F_v^\times \cong \mathcal{O}_v^\times \times \varpi_v^\mathbb{Z}$, where ϖ_v is a uniformiser of \mathcal{O}_v , so that $\varpi_v \mathcal{O}_v = \mathfrak{p}_v$. We can write $\omega_v = \beta_v |\cdot|_v^s$ for some character β_v of \mathcal{O}_v^\times (or rather, some character β_v of F_v^\times that is trivial on $\varpi_v^\mathbb{Z}$) and some $s \in \mathbb{C}$, where the absolute value $|\cdot|_v$ on F_v^\times is normalised such that $|\varpi_v|_v^{-1}$ is equal to $\#\mathcal{O}_v/\mathfrak{p}_v =: q_v$, the order of the residue field of F_v . If ω_v is trivial on \mathcal{O}_v^\times , then ω_v is said to be unramified and of conductor \mathcal{O}_v and conductor exponent $c(\omega_v) = 0$. Otherwise, ω_v is said to be ramified with conductor exponent $c(\omega_v)$ defined to be the least nonnegative integer m for which $\omega_v(1 + \mathfrak{p}_v^m) = 1$. The conductor of ω_v is then defined to be the ideal $\mathfrak{p}_v^{c(\omega_v)}$ of \mathcal{O}_v . The character β_v of \mathcal{O}_v^\times descends to a primitive character of the finite abelian group $\mathcal{O}_v^\times / (1 + \mathfrak{p}_v^{c(\omega_v)}) \cong (\mathcal{O}_F/\mathfrak{p}^{c(\omega_{\mathfrak{p}})}\mathcal{O}_F)^\times$.

Let (π_v, V_v) be an irreducible admissible representation of $\mathrm{GL}_2(F_v)$ with v a nonarchimedean place of F . For this to be a local component of an automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, π_v must be generic. That is, we fix a nontrivial additive character ψ_v of F_v ; then there exists a nonzero linear functional $\Lambda_v: V_v \rightarrow \mathbb{C}$, the local Whittaker functional with respect to ψ_v , that is unique up to multiplication by a scalar and satisfies

$$\Lambda_v \left(\pi_v \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \xi_v \right) = \psi_v(x) \Lambda_v(\xi_v)$$

for every $x \in F_v$ and $\xi_v \in V_v$.

Consider the following vector subspace of V_v :

$$V_v^{\mathrm{GL}_2(\mathcal{O}_v)} := \{ \xi_v \in V_v : \pi_v(g_v) \cdot \xi_v = \xi_v \text{ for all } g_v \in \mathrm{GL}_2(\mathcal{O}_v) \}.$$

This can be shown to be either zero- or one-dimensional. In the latter case, π_v is said to be unramified, the conductor of π_v is defined to be \mathcal{O}_v , and the conductor exponent is $c(\pi_v) = 0$. In the former case, π_v is said to be ramified; nevertheless, Casselman [Cas73] showed that for ramified π_v , there exists some minimal positive integer m such that

$$V_v^{K_0(\mathfrak{p}_v^m), \omega_{\pi_v}} := \left\{ \xi_v \in V_v : \pi_v \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \xi_v = \omega_{\pi_v}(d) \xi_v \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{p}_v^m) \right\}$$

is not equal to $\{0\}$, where ω_{π_v} is the central character of π_v and

$$K_0(\mathfrak{p}_v^m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_v) : c \in \mathfrak{p}_v^m \right\}.$$

This minimal integer m is called the conductor exponent of π_v and denoted $c(\pi_v)$, and the ideal $\mathfrak{p}_v^{c(\pi_v)}$ of \mathcal{O}_v is called the conductor of π_v . Once again, this vector subspace is one-dimensional. In both the unramified and ramified cases, the associated one-dimensional vector subspace contains a unique nonzero element $\xi_v^\circ \in V_v$, the local newvector of π_v , satisfying $\Lambda_v(\xi_v^\circ) = 1$.

When v is archimedean, the analogue of a local newform is a local lowest weight vector ξ_v° of V_v . If v is a real place of F , we let $K_v := \mathrm{O}_2(F_v)$, and we have that a generic irreducible admissible $(\mathfrak{gl}_2(F_v), K_v)$ -module (π_v, V_v) has a smallest nonnegative integer k_v for which there exists a nonzero vector $\xi_v^\circ \in V_v$ satisfying

$$\pi_v \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot \xi_v^\circ = e^{ik_v \theta} \xi_v^\circ$$

for all $\theta \in \mathbb{R}$. When v is complex, and setting $K_v := \mathrm{U}_2(F_v)$, a generic irreducible admissible $(\mathfrak{gl}_2(F_v), K_v)$ -module (π_v, V_v) has a least integer k_v such that the restriction of π_v to K_v contains a representation of dimension $k_v + 1$, and furthermore there exists a nonzero vector $\xi_v^\circ \in V_v$ satisfying

$$\pi_v \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cdot \xi_v^\circ = e^{ik_v \theta} \xi_v^\circ$$

for all $\theta \in \mathbb{R}$. In both cases, the vector ξ_v° is unique up to scalar multiplication. Again, as π_v is generic, it admits a local Whittaker functional Λ_v with respect to a fixed nontrivial additive character ψ_v of F_v , and we may then choose the local lowest weight vector ξ_v° to be such that $\Lambda_v(\xi_v^\circ) = 1$.

Let (π, V) be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of central character $\omega_\pi = \prod_v \omega_{\pi_v}$ on the space V of cuspidal automorphic forms of $\mathrm{GL}_2(\mathbb{A}_F)$ of central character ω_π . Throughout, we will always assume that automorphic representations and Hecke characters are unitary. By Flath's tensor product theorem, there exists an irreducible admissible $(\mathfrak{gl}_2(F_v), K_v)$ -module (π_v, V_v) for each archimedean place v and a generic irreducible admissible unitarisable representation (π_v, V_v) of $\mathrm{GL}_2(F_v)$ with central character ω_{π_v} for each nonarchimedean place v such that (π, V) is isomorphic to the restricted tensor product $(\bigotimes_v \pi_v, \bigotimes_v V_v)$ with the identification according to the local newvector or local lowest weight vector $\xi_v^\circ \in V_v$ at each place. Here each local nontrivial additive character ψ_v is the local component of $\psi = \prod_v \psi_v := \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{\mathbb{A}_F/\mathbb{A}_{\mathbb{Q}}}$, where $\mathrm{Tr}_{\mathbb{A}_F/\mathbb{A}_{\mathbb{Q}}} : \mathbb{A}_F \rightarrow \mathbb{A}_{\mathbb{Q}}$ denotes the global trace map and $\psi_{\mathbb{Q}}$ is the unique additive character on $\mathbb{A}_{\mathbb{Q}}$ that is unramified at every nonarchimedean place and is equal to $e^{2\pi i x}$ at the archimedean place.

We define the arithmetic conductor of π to be the integral ideal

$$\mathfrak{q} := \prod_{v \in S_f} (\mathfrak{p}_v \cap \mathcal{O}_F)^{c(\pi_v)}$$

of \mathcal{O}_F ; the representation π is ramified at only finitely many places, so this is well-defined. Similarly, we define the arithmetic conductor of a Hecke character $\omega = \prod_v \omega_v$ of $F^\times \backslash \mathbb{A}_F^\times$ to be the integral ideal

$$\mathfrak{f} := \prod_{v \in S_f} (\mathfrak{p}_v \cap \mathcal{O}_F)^{c(\omega_v)}.$$

Again, this is well-defined as ω is ramified at only finitely many places. When $\omega = \omega_\pi$ is the central character of π , one can show that $c(\omega_{\pi_v}) \leq c(\pi_v)$ for each nonarchimedean place v of F , so that the arithmetic conductor \mathfrak{f} of ω_π divides the arithmetic conductor \mathfrak{q} of π .

The global newvector of an automorphic representation (π, V) is the cuspidal automorphic form $\xi^\circ \in V$ that corresponds to the pure tensor $\bigotimes_v \xi_v^\circ$ under the isomorphism $(\pi, V) \cong (\bigotimes_v \pi_v, \bigotimes_v V_v)$. When $F = \mathbb{Q}$, the global newvector is the adèlic lift of a classical newform φ . Moreover, the central character ω_π is the idèlic lift of the character ε of φ , while the arithmetic conductor of ω_π is the ideal in \mathbb{Z} generated by the conductor of ε , the arithmetic conductor of π is the ideal generated by the level q of φ , and the archimedean component π_∞ of π specifies its Laplacian eigenvalue λ_φ or its weight k should φ be a Maaß or holomorphic

cuspidal form respectively. Conversely, given a newform φ of level q and character ε , there exists a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, unique up to isomorphism, with conductor exponent $c(\pi_p)$ at each prime p satisfying $p^{c(\pi_p)} \parallel q$, whose central character ω_{π} is the idèlic lift of ε , and whose global newvector ξ° is the adèlic lift of φ . In particular, there is a bijective correspondence between newforms and cuspidal automorphic representations.

2.2. Bounding the Conductor Exponent of a Twist. We define the twist by a Hecke character ω of a cuspidal automorphic representation (π, V) of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character ω_{π} to be the cuspidal automorphic representation $(\pi \otimes \omega, V_{\omega})$ with central character $\omega^2 \omega_{\pi}$ acting on the vector space

$$V_{\omega} := \{\phi_{\omega}(g) := \omega(g)\phi(g) : \phi \in V\}.$$

If we write $\omega = \prod_v \omega_v$, then the local components of $(\pi \otimes \omega, V_{\omega})$ are again generic irreducible admissible representations $(\pi_v \otimes \omega_v, V_v)$ with central character $\omega_v^2 \omega_{\pi_v}$ and action

$$(\pi_v \otimes \omega_v)(g_v) \cdot \xi_v := \omega_v(\det g_v) \pi_v(g_v) \cdot \xi_v$$

for $g_v \in \mathrm{GL}_2(F_v)$ and $\xi_v \in V_v$.

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, and suppose that there exists a twist $\pi \otimes \omega$ of π by some Hecke character ω such that $\pi \otimes \omega$ has the same central character and arithmetic conductor as π . Then ω must be a quadratic character, and the local conductor exponents must satisfy $c(\pi_v \otimes \omega_v) = c(\pi_v)$ at every nonarchimedean place v . If ω_v is unramified, then this equality automatically holds. On the other hand, if ω_v is ramified, then this equality is not always ensured. We will determine when this occurs in the particular case when the central character of π is trivial. The result is obtained on a case-by-case analysis, which requires the classification of generic irreducible admissible representations of $\mathrm{GL}_2(F_v)$. A key tool is the following bound for the conductor exponent.

Lemma 2.1. *Let v be a nonarchimedean place of an algebraic number field F . Let (π_v, V_v) be a generic irreducible admissible representation of $\mathrm{GL}_2(F_v)$ with central character ω_{π_v} , and let ω_v be a unitary character of F_v^{\times} . Then we have the following bound for the conductor exponent of $\pi_v \otimes \omega_v$:*

$$c(\pi_v \otimes \omega_v) \leq \max\{c(\pi_v), c(\omega_v) + c(\omega_{\pi_v}), 2c(\omega_v)\}.$$

This is proved in [AL78, Proposition 3.1] by Atkin and Li in the classical setting of holomorphic newforms (that is, cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ whose archimedean component is a discrete series representation or a limit of discrete series representation). We give a local proof here that essentially mimics Atkin and Li's proof.

Proof. Let $\xi_v^{\circ} \in V_v$ be the local newvector of π_v with respect to the local Whittaker functional Λ_v corresponding to a fixed nontrivial unramified additive character ψ_v of F_v . We define $\xi'_v \in V_v$ by

$$\xi'_v := \tau(\overline{\omega_v})^{-1} \sum_{u \in \mathcal{O}_v^{\times} / (1 + \mathfrak{p}_v^{c(\omega_v)})} \overline{\omega_v}(u) \pi_v \begin{pmatrix} 1 & u \overline{\omega_v}^{-c(\omega_v)} \\ 0 & 1 \end{pmatrix} \cdot \xi_v^{\circ},$$

where

$$\tau(\overline{\omega_v}) := \sum_{u \in \mathcal{O}_v^{\times} / (1 + \mathfrak{p}_v^{c(\omega_v)})} \overline{\omega_v}(u) \psi_v(u \overline{\omega_v}^{-c(\omega_v)})$$

is the Gauss sum of the primitive multiplicative character $\overline{\omega}_v$ and the unramified additive character ψ_v , and in particular is nonzero. The vector ξ'_v is well-defined, as if $u, u' \in \mathcal{O}_v^\times$ with $u - u' \in \mathfrak{p}_v^{c(\omega_v)}$, then

$$\begin{aligned} \pi_v \begin{pmatrix} 1 & u\overline{\omega}_v^{-c(\omega_v)} \\ 0 & 1 \end{pmatrix} \cdot \xi_v &= \pi_v \begin{pmatrix} 1 & u'\overline{\omega}_v^{-c(\omega_v)} \\ 0 & 1 \end{pmatrix} \cdot \left(\pi_v \begin{pmatrix} 1 & 1 + (u - u')\overline{\omega}_v^{-c(\omega_v)} \\ 0 & 1 \end{pmatrix} \cdot \xi_v^\circ \right) \\ &= \pi_v \begin{pmatrix} 1 & u'\overline{\omega}_v^{-c(\omega_v)} \\ 0 & 1 \end{pmatrix} \cdot \xi_v^\circ \end{aligned}$$

by the fact that $\xi_v^\circ \in V_v^{K_0(\mathfrak{p}_v^{c(\pi_v)}), \omega_{\pi_v}}$. Furthermore, the vector ξ'_v is nonzero, as can be seen from the equality $\Lambda_v(\xi'_v) = \Lambda_v(\xi_v^\circ) = 1$, which is a consequence of the definition of ξ'_v and the fact that ξ_v° is a local newvector.

We must show that $\xi'_v \in V_v^{K_0(\mathfrak{p}_v^m), \omega_{\pi_v}^2}$, where

$$m = \max \{c(\pi_v), c(\omega_v) + c(\omega_{\pi_v}), 2c(\omega_v)\}.$$

We first note that for each $u \in \mathcal{O}_v^\times / (1 + \mathfrak{p}_v^{c(\omega_v)})$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{p}_v^m)$, so that $a, d \in \mathcal{O}_v^\times$, there exists a unique $u' \in \mathcal{O}_v^\times / (1 + \mathfrak{p}_v^{c(\omega_v)})$ for which $ud - u'a \in \mathfrak{p}_v^{c(\omega_v)}$. Moreover, we have that

$$\begin{pmatrix} 1 & u\overline{\omega}_v^{-c(\omega_v)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 & u'\overline{\omega}_v^{-c(\omega_v)} \\ 0 & 1 \end{pmatrix}$$

where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + uc\overline{\omega}_v^{-c(\omega_v)} & b + (ud - u'a)\overline{\omega}_v^{-c(\omega_v)} - uu'\overline{\omega}_v^{-2c(\omega_v)} \\ c & d - u'c\overline{\omega}_v^{-c(\omega_v)} \end{pmatrix}.$$

This is in $K_0(\mathfrak{p}_v^m)$ by the fact that $c \in \mathfrak{p}_v^m$ with $m \geq 2c(\omega_v)$ and $ud - u'a \in \mathfrak{p}_v^{c(\omega_v)}$. It follows that

$$\begin{aligned} (\pi_v \otimes \omega_v) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \xi'_v &= \omega_v(ad - bc)\tau(\overline{\omega}_v)^{-1} \\ &\times \sum_{u' \in \mathcal{O}_v^\times / (1 + \mathfrak{p}_v^{c(\omega_v)})} \overline{\omega}_v(u'ad^{-1})\pi_v \left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 & u'\overline{\omega}_v^{-c(\omega_v)} \\ 0 & 1 \end{pmatrix} \right) \cdot \xi_v^\circ. \end{aligned}$$

Now $m \geq c(\pi_v)$, so that $\xi_v^\circ \in V_v^{K_0(\mathfrak{p}_v^m), \omega_{\pi_v}}$, so this is

$$\omega_v(d^2 - a^{-1}bc)\tau(\overline{\omega}_v)^{-1} \sum_{u' \in \mathcal{O}_v^\times / (1 + \mathfrak{p}_v^{c(\omega_v)})} \overline{\omega}_v(u')\omega_{\pi_v}(d')\pi_v \begin{pmatrix} 1 & u'\overline{\omega}_v^{-c(\omega_v)} \\ 0 & 1 \end{pmatrix} \cdot \xi_v^\circ.$$

As $m \geq c(\omega_v) + c(\omega_{\pi_v})$, so that $d' - d \in \mathfrak{p}_v^{c(\omega_{\pi_v})}$ and $a^{-1}bc \in \mathfrak{p}_v^{c(\omega_v)}$, we obtain

$$(\pi_v \otimes \omega_v) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \xi'_v = \omega_v(d)^2\omega_{\pi_v}(d)\xi'_v. \quad \square$$

2.3. Representations of $\mathrm{GL}_2(F_v)$ at Nonarchimedean Places. Let v be a nondyadic nonarchimedean place of F , which is to say that $\mathfrak{p}_v \cap \mathbb{Z} \neq 2\mathbb{Z}$. There exists a unique nontrivial character β_v of \mathcal{O}_v^\times satisfying $\beta_v^2 = 1$, which we denote $\beta_{v,\text{quad}}$. We may view $\beta_{v,\text{quad}}$ as a character of F_v^\times that is trivial on $\overline{\omega}_v^{\mathbb{Z}}$; as v is nondyadic, $\beta_{v,\text{quad}}$ has conductor \mathfrak{p}_v (and hence conductor exponent 1). As every character ω_v of F_v^\times is equal to $\beta_v \cdot |\cdot|_v^s$ for some character β_v of \mathcal{O}_v^\times and some $s \in \mathbb{C}$, it follows that there are three quadratic characters of F_v^\times , namely

$$|\cdot|_v^{\frac{\pi i}{\log q_v}}, \quad \beta_{v,\text{quad}}, \quad \beta_{v,\text{quad}}|\cdot|_v^{\frac{\pi i}{\log q_v}}.$$

The first character is unramified, and so twisting a representation π_v by $|\cdot|_v^{\frac{\pi i}{\log q_v}}$ does not change the conductor of π_v , while for the third, we have that

$$\pi_v \otimes \beta_{v,\text{quad}} |\cdot|_v^{\frac{\pi i}{\log q_v}} = \left(\pi_v \otimes |\cdot|_v^{\frac{\pi i}{\log q_v}} \right) \otimes \beta_{v,\text{quad}}.$$

So it suffices to classify the representations π_v for which $c(\pi_v \otimes \beta_{v,\text{quad}}) = c(\pi_v)$.

Let π_v be a generic irreducible admissible unitarisable representation of $\text{GL}_2(F_v)$ with central character ω_v . We recall that such representations can be classified as either principal series representations, special representations, or supercuspidal representations.

2.3.1. Principal Series Representations of $\text{GL}_2(F_v)$. Let F_v be a local field with maximal compact subgroup K_v , and let $\omega_{v,1}, \omega_{v,2}$ be characters of F_v^\times . A principal series representation

$$\pi_v \cong \omega_{v,1} \boxplus \omega_{v,2}$$

of $\text{GL}_2(F_v)$ is the right regular representation of $\text{GL}_2(F_v)$ on the space of functions $f: \text{GL}_2(F_v) \rightarrow \mathbb{C}$ that are square-integrable with respect to the inner product

$$\langle f_1, f_2 \rangle := \int_{K_v} f_1(k_v) \overline{f_2(k_v)} dk_v$$

and satisfy

$$f \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g_v \right) = \omega_{v,1}(a) \omega_{v,2}(d) \left| \frac{a}{d} \right|_v^{1/2} f(g_v)$$

for all $a, d \in F_v^\times$, $cb \in F_v$, and $g_v \in \text{GL}_2(F_v)$.

When v is nonarchimedean, such representations are irreducible and unitarisable if the characters $\omega_{v,1}, \omega_{v,2}$ are of the form

$$\omega_{v,1} = \beta_{v,1} |\cdot|_v^{s_1}, \quad \omega_{v,2} = \beta_{v,2} |\cdot|_v^{s_2}$$

with $\beta_{v,1}, \beta_{v,2}$ characters of \mathcal{O}_v^\times , and either $s_1, s_2 \in i\mathbb{R}$, or $s_1 + s_2 \in i\mathbb{R}$ with $s_1 - s_2 \in (-1, 1)$ and $\beta_{v,1} = \beta_{v,2}$. The central character of π_v is

$$\omega_{\pi_v} = \omega_{v,1} \omega_{v,2} = \beta_{v,1} \beta_{v,2} |\cdot|_v^{s_1 + s_2}.$$

The only other irreducible representation of $\text{GL}_2(F_v)$ isomorphic to $\omega_{v,1} \boxplus \omega_{v,2}$ is $\omega_{v,2} \boxplus \omega_{v,1}$. The conductor exponent $c(\pi_v)$ of π_v is $c(\omega_{v,1}) + c(\omega_{v,2})$. If ω'_v is a character of F_v^\times , the twist of π_v by ω'_v is the principal series representation

$$\pi_v \otimes \omega'_v \cong \omega_{v,1} \omega'_v \boxplus \omega_{v,2} \omega'_v.$$

When π_v has trivial central character, we have that $\omega_{v,2} = \omega_{v,1}^{-1}$, so that $\beta_{v,2} = \beta_{v,1}^{-1}$ and $s_2 = -s_1$, and that $c(\pi_v) = 2c(\omega_{v,1})$; in particular, the conductor exponent must be even in this case.

Lemma 2.2. *Let π_v be a ramified irreducible unitarisable principal series representation with trivial central character, so that*

$$\pi_v \cong \beta_v |\cdot|_v^s \boxplus \beta_v^{-1} |\cdot|_v^{-s}$$

for some ramified character β_v of \mathcal{O}_v^\times and some $s \in i\mathbb{R}$. Then we have that

$$c(\pi_v \otimes \beta_{v,\text{quad}}) = c(\pi_v)$$

unless $\beta_v = \beta_{v,\text{quad}}$, in which case $c(\pi_v) = 2$ but $c(\pi_v \otimes \beta_{v,\text{quad}}) = 0$.

Proof. The conductor exponent $c(\pi_v)$ of π_v is $2c(\beta_v)$, and in particular is positive and even. We have that

$$\pi_v \otimes \beta_{v,\text{quad}} \cong \beta_v \beta_{v,\text{quad}} |\cdot|_v^s \boxplus \beta_v^{-1} \beta_{v,\text{quad}} |\cdot|_v^{-s},$$

which also has trivial central character.

Now if $\beta_v = \beta_{v,\text{quad}}$, in which case $c(\pi_v) = 2$, we have that

$$\pi_v \otimes \beta_{v,\text{quad}} = |\cdot|_v^s \boxplus |\cdot|_v^{-s},$$

and in particular $c(\pi_v \otimes \beta_{v,\text{quad}}) = 0$. So π_v is the twist by $\beta_{v,\text{quad}}$ of the unramified principal series representation $|\cdot|_v^s \boxplus |\cdot|_v^{-s}$ with conductor exponent 0.

If $\beta_v \neq \beta_{v,\text{quad}}$, we prove by strong induction that $c(\pi_v \otimes \beta_{v,\text{quad}}) = c(\pi_v)$. The base case is $c(\pi_v) = 2$. Then by Lemma 2.1,

$$c(\pi_v \otimes \beta_{v,\text{quad}}) \leq \max\{c(\pi_v), c(\beta_{v,\text{quad}}) + c(\omega_{\pi_v}), 2c(\beta_{v,\text{quad}})\} = 2,$$

but as $\pi_v \otimes \beta_{v,\text{quad}}$ has trivial central character and is not unramified, we have that $c(\pi_v \otimes \beta_{v,\text{quad}}) \geq 2$, and so equality holds. For the strong induction, we observe that again

$$c(\pi_v \otimes \beta_{v,\text{quad}}) \leq \max\{c(\pi_v), c(\beta_{v,\text{quad}}) + c(\omega_{\pi_v}), 2c(\beta_{v,\text{quad}})\} = c(\pi_v)$$

by Lemma 2.1, but that $c(\pi_v \otimes \beta_{v,\text{quad}}) \geq c(\pi_v)$ by the strong induction hypothesis, for otherwise we could twist by $\beta_{v,\text{quad}}$ once again. This yields the result. \square

2.3.2. Special Representations of $\text{GL}_2(F_v)$. A special representation is the unique irreducible subrepresentation

$$\pi_v \cong \omega_v \text{St}_v$$

of codimension one of the reducible principal series representation $\omega_v|\cdot|_v^{1/2} \boxplus \omega_v|\cdot|_v^{-1/2}$, where $\omega_v = \beta_v|\cdot|_v^s$ is a character of F_v^\times , with β_v a character of \mathcal{O}_v^\times and $s \in i\mathbb{R}$. The central character of π_v is

$$\omega_{\pi_v} = \omega_v^2 = \beta_v^2|\cdot|_v^{2s}.$$

There is no other irreducible representation of $\text{GL}_2(F_v)$ isomorphic to $\omega_v \text{St}_v$. The conductor exponent is

$$c(\pi_v) = \begin{cases} 1 & \text{if } c(\omega_v) = 0, \\ 2c(\omega_v) & \text{otherwise.} \end{cases}$$

If ω'_v is a character of F_v^\times , the twist of π_v by ω'_v is the special representation

$$\pi_v \otimes \omega'_v = \omega_v \omega'_v \text{St}_v.$$

When π_v has trivial central character, we have that $\omega_v^2 = 1$; consequently, we must have that $s_1 = s_{\text{quad}} \in \left\{0, \frac{\pi i}{\log q_v}\right\}$ and $\beta_v \in \{1, \beta_{v,\text{quad}}\}$, and that

$$c(\pi_v) = \begin{cases} 1 & \text{if } \beta_v = 1, \\ 2 & \text{if } \beta_v = \beta_{v,\text{quad}}. \end{cases}$$

Lemma 2.3. *Let $\pi_v \cong \omega_v \text{St}_v$ be a special representation with trivial central character, so that $\omega_v = \beta_v|\cdot|_v^{s_{\text{quad}}}$ with $\beta_v \in \{1, \beta_{v,\text{quad}}\}$ and $s_{\text{quad}} \in \left\{0, \frac{\pi i}{\log q_v}\right\}$. Then*

$$\pi_v \otimes \beta_{v,\text{quad}} \cong \begin{cases} \beta_{v,\text{quad}}|\cdot|_v^{s_{\text{quad}}} \text{St}_v & \text{if } \beta_v = 1, \\ |\cdot|_v^{s_{\text{quad}}} \text{St}_v & \text{if } \beta_v = \beta_{v,\text{quad}}, \end{cases}$$

and hence

$$c(\pi_v \otimes \beta_{v,\text{quad}}) = \begin{cases} c(\pi_v) + 1 & \text{if } \beta_v = 1, \\ c(\pi_v) - 1 & \text{if } \beta_v = \beta_{v,\text{quad}}. \end{cases}$$

Proof. We have that

$$c(\pi_v) = \begin{cases} 1 & \text{if } \beta_v = 1, \\ 2 & \text{if } \beta_v = \beta_{v,\text{quad}}. \end{cases}$$

As $\pi_v \otimes \beta_{v,\text{quad}} \cong \omega_v \beta_{v,\text{quad}} \text{St}_v$, it follows that if $\beta_v = \beta_{v,\text{quad}}$, then π_v is the twist by $\beta_{v,\text{quad}}$ of the special representation $|\cdot|_v^{s_{\text{quad}}} \text{St}_v$ with conductor \mathfrak{p}_v . Similarly, if

$\beta_v = 1$, then π_v is the twist by $\beta_{v,\text{quad}}$ of the special representation $\beta_{v,\text{quad}}|\cdot|_v^{s_{\text{quad}}}\text{St}_v$ with conductor \mathfrak{p}_v^2 . In particular,

$$c(\pi_v \otimes \beta_{v,\text{quad}}) = \begin{cases} 2 & \text{if } \beta_v = 1, \\ 1 & \text{if } \beta_v = \beta_{v,\text{quad}}. \end{cases} \quad \square$$

Lemma 2.4. *Every generic irreducible admissible unitarisable representation of $\text{GL}_2(F_v)$ with conductor \mathfrak{p}_v and trivial central character is of the form $|\cdot|_v^{s_{\text{quad}}}\text{St}_v$ with $s_{\text{quad}} \in \left\{0, \frac{\pi i}{\log q_v}\right\}$.*

Proof. Such a representation cannot be a principal series representation, as the central character is trivial but the conductor exponent is not even, or a supercuspidal representation, as the conductor exponent is less than 2. It remains to note that every special representation of conductor \mathfrak{p}_v and trivial central character ω_v^2 is of the desired form. \square

2.3.3. Supercuspidal Representations of $\text{GL}_2(F_v)$. Supercuspidal representations are the compact induction to $\text{GL}_2(F_v)$ of a finite-dimensional representation ρ_{π_v} of a subgroup H of $\text{GL}_2(F_v)$ that is compact modulo the centre $Z(F_v)$ of $\text{GL}_2(F_v)$. Every maximal open subgroup of $\text{GL}_2(F_v)$ that is compact modulo the centre is conjugate to either $Z(F_v)\text{GL}_2(\mathcal{O}_v)$ or $NK_0(\mathfrak{p}_v)$, the normaliser of $K_0(\mathfrak{p}_v)$ in $\text{GL}_2(F_v)$. We say that a supercuspidal representation π_v is of type I if H is conjugate to $Z(F_v)\text{GL}_2(\mathcal{O}_v)$ and of type II if H is conjugate to $NK_0(\mathfrak{p}_v)$. These representations always have conductor exponent $c(\pi_v)$ at least 2. The twist $\pi_v \otimes \omega'_v$ of a supercuspidal representation π_v by a character ω'_v of F_v^\times is also a supercuspidal representation.

Lemma 2.5. *Let π_v be a supercuspidal representation with trivial central character. Then the twist $\pi_v \otimes \beta_{v,\text{quad}}$ is a supercuspidal representation with trivial central character and conductor exponent $c(\pi_v \otimes \beta_{v,\text{quad}}) = c(\pi_v)$.*

Proof. It is clear that $\pi_v \otimes \beta_{v,\text{quad}}$ is a supercuspidal representation with trivial central character. The fact that the conductor remains unchanged follows by strong induction as in the proof of Lemma 2.2, with the base case being $c(\pi_v) = 2$ by the fact that every supercuspidal representation has conductor exponent at least 2. \square

2.4. $(\mathfrak{gl}_2(F_v), K_v)$ -Modules. Let $S_{\mathbb{R}}$ and $S_{\mathbb{C}}$ denote the real and complex places of F respectively. For $v \in S_{\infty}$, a generic irreducible admissible unitarisable $(\mathfrak{gl}_2(F_v), K_v)$ -module π_v corresponds to an infinitesimal equivalence class of principal series representations or discrete series representations, with the latter only possible if $v \in S_{\mathbb{R}}$.

2.4.1. Principal Series Representations of $\text{GL}_2(\mathbb{R})$. Let v be a real place of F . Principal series representations of $\text{GL}_2(F_v)$ are of the form

$$\pi_v \cong \omega_{v,1} \boxplus \omega_{v,2}$$

for some characters $\omega_{v,1} = \text{sgn}_v^{m_1} |\cdot|_v^{s_1}$, $\omega_{v,2} = \text{sgn}_v^{m_2} |\cdot|_v^{s_2}$ of \mathbb{R}^\times , with $m_1, m_2 \in \mathbb{Z}/2\mathbb{Z}$ and either $s_1, s_2 \in i\mathbb{R}$, or $s_1 + s_2 \in i\mathbb{R}$ with $s_1 - s_2 \in (-1, 1)$ and $m_1 = m_2$. Here $\text{sgn}_v(x) := x/|x|_v$ is the sign of an element $x \in F_v$, with $|x|_v := \max\{x, -x\}$ the usual absolute value on \mathbb{R} . The central character of π_v is

$$\omega_{\pi_v} = \omega_{v,1}\omega_{v,2} = \text{sgn}_v^{m_1+m_2} |\cdot|_v^{s_1+s_2}.$$

The only other irreducible representation of $\text{GL}_2(F_v)$ isomorphic to $\omega_{v,1} \boxplus \omega_{v,2}$ is $\omega_{v,2} \boxplus \omega_{v,1}$. If ω'_v is a character of \mathbb{R}^\times , the twist of π_v by ω'_v is the principal series representation

$$\pi_v \otimes \omega'_v \cong \omega_{v,1}\omega'_v \boxplus \omega_{v,2}\omega'_v.$$

We define the weight $k_v \in \{0, 1\}$ and spectral parameter $s_v \in i\mathbb{R} \cup (-1/2, 1/2)$ of π_v to be

$$k_v := |m_1 - m_2|, \quad s_v := \frac{s_1 - s_2}{2};$$

note that the latter is defined up to multiplication by ± 1 . The quadratic twist of π_v by the unique quadratic character sgn_v of \mathbb{R}^\times merely sends (m_1, m_2) to $(m_1 + 1, m_2 + 1)$ in $\mathbb{Z}/2\mathbb{Z}$ while leaving s_1, s_2 unaltered. Thus the weight k_v and spectral parameter s_v remain unchanged. When π_v has trivial central character, we have that $\omega_{v,2} = \omega_{v,1}^{-1}$, so that $m_2 = m_1$ and $s_2 = -s_1$. In this case, we have that $k_v = 0$ and $s_v = s_1 \in i\mathbb{R} \cup (-1/2, 1/2)$.

2.4.2. Discrete Series Representations of $\text{GL}_2(\mathbb{R})$. A discrete series representation of $\text{GL}_2(F_v)$, $v \in S_\mathbb{R}$, is the unique irreducible subrepresentation

$$\pi_v \cong \mathcal{D}(\omega_{v,1}, \omega_{v,2})$$

of codimension one of the reducible principal series representation $\omega_{v,1} \boxplus \omega_{v,2}$ for some characters $\omega_{v,1} = \text{sgn}_v^{m_1} \cdot |\cdot|_v^{s_1}$, $\omega_{v,2} = \text{sgn}_v^{m_2} \cdot |\cdot|_v^{s_2}$ of \mathbb{R}^\times with $m_1, m_2 \in \mathbb{Z}/2\mathbb{Z}$, and $s_1, s_2 \in \mathbb{C}$ such that $s_1 + s_2 \in i\mathbb{R}$, $s_1 - s_2 \in \mathbb{Z} \setminus \{0\}$, and $s_1 - s_2 + 1 \equiv m_1 - m_2 \pmod{2}$. The other irreducible representations of $\text{GL}_2(F_v)$ isomorphic to $\mathcal{D}(\omega_{v,1}, \omega_{v,2})$ are $\mathcal{D}(\omega_{v,2}, \omega_{v,1})$, $\mathcal{D}(\text{sgn}_v \omega_{v,1}, \text{sgn}_v \omega_{v,2})$, and $\mathcal{D}(\text{sgn}_v \omega_{v,2}, \text{sgn}_v \omega_{v,1})$. The central character of π_v is again

$$\omega_{\pi_v} = \omega_{v,1} \omega_{v,2} = \text{sgn}_v^{m_1+m_2} \cdot |\cdot|_v^{s_1+s_2}.$$

If ω'_v is a character of \mathbb{R}^\times , the twist of π_v by ω'_v is the discrete series representation

$$\pi_v \otimes \omega'_v \cong \mathcal{D}(\omega_{v,1}\omega'_v, \omega_{v,2}\omega'_v).$$

We define the weight $k_v \in \mathbb{N} \setminus \{0\}$ and spectral parameter s_v of π_v to be

$$k_v := |s_1 - s_2| + 1, \quad s_v := \frac{s_1 - s_2}{2},$$

with the latter defined up to multiplication by ± 1 . The quadratic twist of π_v by sgn_v is isomorphic to π_v , and in particular leaves the weight k_v and spectral parameter s_v unchanged. Again, when π_v has trivial central character, we have that $\omega_{v,2} = \omega_{v,1}^{-1}$, so that $m_2 = m_1$ and $s_2 = -s_1$. In this case, we have that $k_v = 2|s_1| + 1 \in 2\mathbb{N}$ and $s_v = s_1 \in 1/2 + \mathbb{Z}$.

2.4.3. Principal Series Representations of $\text{GL}_2(\mathbb{C})$. A principal series representation of $\text{GL}_2(F_v)$ with $v \in S_\mathbb{C}$ is of the form $\pi_v \cong \omega_{v,1} \boxplus \omega_{v,2}$ for some characters $\omega_{v,1} = e^{im_1 \arg_v} \cdot |\cdot|_v^{s_1}$, $\omega_{v,2} = e^{im_2 \arg_v} \cdot |\cdot|_v^{s_2}$ of \mathbb{C}^\times , with $m_1, m_2 \in \mathbb{Z}$ and either $s_1, s_2 \in i\mathbb{R}$, or $s_1 + s_2 \in i\mathbb{R}$ with $s_1 - s_2 \in (-1, 1)$ and $m_1 = m_2$. Here $\arg_v(z)$ is the argument of an element $z \in F_v$, so that $e^{i \arg_v(z)} := z/|z|_v^{1/2}$ with $|z|_v := z\bar{z}$ the square of the usual absolute value on \mathbb{C} . The only other irreducible representation of $\text{GL}_2(F_v)$ isomorphic to $\omega_{v,1} \boxplus \omega_{v,2}$ is $\omega_{v,2} \boxplus \omega_{v,1}$. If ω'_v is a character of \mathbb{R}^\times , the twist of π_v by ω'_v is the principal series representation

$$\pi_v \otimes \omega'_v \cong \omega_{v,1}\omega'_v \boxplus \omega_{v,2}\omega'_v.$$

We again define the weight $k_v \in \mathbb{N} \cup \{0\}$ and spectral parameter $s_v \in i\mathbb{R} \cup (-1/2, 1/2)$ of π_v to be

$$k_v := |m_1 - m_2|, \quad s_v := \frac{s_1 - s_2}{2},$$

with the latter defined up to multiplication by ± 1 . Note that there is no quadratic character of \mathbb{C}^\times . When π_v has trivial central character, we have that $\omega_{v,2} = \omega_{v,1}^{-1}$, so that $m_2 = -m_1$ and $s_2 = -s_1$. In this case, we have that $k_v = 2|m_1| \in 2\mathbb{N}$ and $s_v = s_1 \in i\mathbb{R} \cup (-1/2, 1/2)$.

3. PROOF OF THEOREM 1.2

A cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ is said to have archimedean weights and spectral parameters $(\mathbf{k}, \mathbf{s}) = (k_v, s_v)_{v \in S_\infty}$ if at each archimedean place v of F , the local component π_v of π has weight k_v and spectral parameter s_v . We let $\mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s})$ denote the vector subspace of automorphic forms spanned by the set of global newvectors of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ of arithmetic conductor \mathfrak{q} , trivial central character, and archimedean weights and spectral parameters (\mathbf{k}, \mathbf{s}) .

Theorem 3.1. *Let \mathfrak{q} be an integral ideal of \mathcal{O}_F , and let \mathfrak{p} be a nondyadic prime ideal of \mathcal{O}_F dividing \mathfrak{q} . Let $\omega_{\mathrm{quad}(\mathfrak{p})}$ denote a quadratic Hecke character of $F^\times \backslash \mathbb{A}_F^\times$ that is ramified only at \mathfrak{p} . Then if $\mathfrak{p} \parallel \mathfrak{q}$,*

$$(2) \quad \mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \cap (\mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \otimes \omega_{\mathrm{quad}(\mathfrak{p})}) = \{0\}.$$

If $\mathfrak{p}^2 \parallel \mathfrak{q}$,

$$(3) \quad \begin{aligned} \mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \cap (\mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \otimes \omega_{\mathrm{quad}(\mathfrak{p})}) \\ = \mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \ominus (\mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}\mathfrak{p}^{-1}), \mathbf{k}, \mathbf{s}) \otimes \omega_{\mathrm{quad}(\mathfrak{p})}) \\ \ominus (\mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}\mathfrak{p}^{-2}), \mathbf{k}, \mathbf{s}) \otimes \omega_{\mathrm{quad}(\mathfrak{p})}). \end{aligned}$$

If $\mathfrak{p}^3 \mid \mathfrak{q}$,

$$(4) \quad \mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \cap (\mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \otimes \omega_{\mathrm{quad}(\mathfrak{p})}) = \mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}).$$

Here the inner product on the space of cuspidal automorphic forms of $\mathrm{GL}_2(\mathbb{A}_F)$ is defined by

$$\langle \phi, \phi' \rangle := \frac{1}{\mathrm{vol}(Z(\mathbb{A}_F) \backslash \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F))} \int_{Z(\mathbb{A}_F) \backslash \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \phi(g) \overline{\phi'}(g) dg,$$

where $Z(\mathbb{A}_F)$ denotes the centre of $\mathrm{GL}_2(\mathbb{A}_F)$; this is the adèlic analogue of the Petersson inner product. A direct consequence of the Rankin–Selberg method is the fact that the global newvectors ϕ, ϕ' of nonisomorphic cuspidal automorphic representations π, π' are orthogonal with respect to this inner product.

The special case $F = \mathbb{Q}$ and $k_\infty = 0$ of Theorem 3.1 is Theorem 1.2 by the correspondence between newforms and cuspidal automorphic representations, with the archimedean weight and spectral parameter (k_∞, s_∞) specifying the weight $k_\infty = 0$ and the Laplacian eigenvalue $\lambda = 1/4 - s_\infty^2$ of the associated newform. When $F = \mathbb{Q}$ and $k_\infty \in 2\mathbb{N}$, on the other hand, Theorem 3.1 reduces to a result of Atkin and Lehner [AL70, Theorem 6] that characterises the level of a holomorphic newform when twisted by a quadratic character of odd prime conductor; their proof is via classical methods, and with a little effort can be modified to prove the analogous result for Maaß newforms, namely Theorem 1.2. On the other hand, such minor modifications do not seem to be possible to extend their work to remain valid for modular forms over an arbitrary number field F , as in Theorem 3.1.

Proof of Theorem 3.1. Let π be a cuspidal automorphic representation associated to a global newvector in $\mathcal{M}^{\mathrm{new}}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s})$. As the archimedean weights and spectral parameters of π are fixed under twisting by a quadratic character, it suffices to determine when $c(\pi_v \otimes \beta_{v, \mathrm{quad}}) = c(\pi_v)$, where v is the nondyadic nonarchimedean place for which $\mathfrak{p}_v \cap \mathcal{O}_F = \mathfrak{p}$ and π_v is the local component of π at v .

If $\mathfrak{p} \parallel \mathfrak{q}$, the local component π_v is a special representation $|\cdot|_v^{s_{\mathrm{quad}}} \mathrm{St}_v$ of conductor \mathfrak{p}_v by Lemma 2.4, and its twist by $\beta_{v, \mathrm{quad}}$ has conductor \mathfrak{p}_v^2 by Lemma 2.3. This proves (2).

If $\mathfrak{p}^3 \mid \mathfrak{q}$, the local component π_v is either a supercuspidal representation or a principal series representation $\beta_v |\cdot|_v^s \boxplus \beta_v^{-1} |\cdot|_v^{-s}$ with $\beta_v^2 \neq 1$, with the latter case only

possible when $\mathfrak{p}^m \parallel \mathfrak{q}$ with $m \in 2\mathbb{N}$. In either case, we have that $c(\pi_v \otimes \beta_{v,\text{quad}}) = c(\pi_v)$ by Lemmata 2.5 and 2.2, from which (4) follows.

If $\mathfrak{p}^2 \parallel \mathfrak{q}$, the local component π_v can either be a principal series representation, a special representation, or a supercuspidal representation. If π_v is a principal series representation, then either $\pi_v \cong \beta_{v,\text{quad}} | \cdot |_v^s \boxplus \beta_{v,\text{quad}} | \cdot |_v^{-s}$ or $\pi_v \cong \beta_v | \cdot |_v^s \boxplus \beta_v^{-1} | \cdot |_v^{-s}$ for some β_v for which $c(\beta_v) = 1$ but $\beta_v^2 \neq 1$. By Lemma 2.2, the representation π_v is the twist by $\beta_{v,\text{quad}}$ of the unramified spherical representation $| \cdot |_v^s \boxplus | \cdot |_v^{-s}$ in the former case, while in the latter case, the conductor of π_v remains \mathfrak{p}_v^2 upon twisting by $\beta_{v,\text{quad}}$. If π_v is a special representation, then π_v is the twist by $\beta_{v,\text{quad}}$ of a special representation $| \cdot |_v^{s_{\text{quad}}} \text{St}_v$ of conductor \mathfrak{p}_v and trivial central character by Lemma 2.3. Finally, when π_v is a supercuspidal representation of conductor \mathfrak{p}_v^2 , Lemma 2.5 shows that the twist of π_v by $\beta_{v,\text{quad}}$ also has conductor \mathfrak{p}_v^2 . The identity (3) then follows. \square

We can extend this result to classify cuspidal automorphic representations having trivial central character that are conductor-invariant under twisting by a quadratic Hecke character of composite arithmetic conductor, albeit with the restriction that the quadratic character is unramified at every dyadic place. We let $\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s})$ denote the set of isomorphism classes of cuspidal automorphic representations of $\text{GL}_2(\mathbb{A}_F)$ of arithmetic conductor \mathfrak{q} , trivial central character, and archimedean weights and spectral parameters (\mathbf{k}, \mathbf{s}) .

Theorem 3.2. *Let \mathfrak{q} and \mathfrak{q}' be integral ideals of \mathcal{O}_F such that \mathfrak{q}' is squarefree and every prime ideal \mathfrak{p} dividing \mathfrak{q}' is non-dyadic. Let $\omega_{\text{quad}(\mathfrak{q}')}$ be a quadratic Hecke character of arithmetic conductor \mathfrak{q}' . Then if there exists a prime ideal \mathfrak{p} dividing \mathfrak{q}' such that \mathfrak{p}^2 does not divide \mathfrak{q} , we have that*

$$(\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \otimes \omega_{\text{quad}(\mathfrak{q}')) \cap \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) = \emptyset,$$

while if \mathfrak{p}^2 divides \mathfrak{q} for every prime ideal \mathfrak{p} dividing \mathfrak{q}' ,

$$\begin{aligned} & \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \cap (\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \otimes \omega_{\text{quad}(\mathfrak{q}')) \\ &= \left\{ \pi \in \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) : \text{for all } v \in S_f \text{ with } \mathfrak{p}_v \cap \mathcal{O}_F = \mathfrak{p} \mid \mathfrak{q}' \text{ and } \mathfrak{p}^2 \parallel \mathfrak{q}, \right. \\ & \quad \left. \pi_v \not\cong \omega_v \boxplus \omega_v, \pi_v \not\cong \omega_v \text{St}_v \text{ with } \omega_v \in \left\{ \beta_{v,\text{quad}}, \beta_{v,\text{quad}} | \cdot |_v^{\frac{\pi i}{\log q_v}} \right\} \right\}. \end{aligned}$$

Moreover, if \mathfrak{q}^* is a prime ideal of \mathcal{O}_F dividing \mathfrak{q}' and $\omega_{\text{quad}(\mathfrak{q}^*)}$ is a quadratic Hecke character of arithmetic conductor \mathfrak{q}^* , then

$$\begin{aligned} & (\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \otimes \omega_{\text{quad}(\mathfrak{q}^*)}) \cap \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \\ & \supset (\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s}) \otimes \omega_{\text{quad}(\mathfrak{q}')} \cap \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, \mathbf{s})). \end{aligned}$$

Proof. This follows directly from the proof of Theorem 3.1 by classifying when $c(\pi_v \otimes \beta_{v,\text{quad}}) = c(\pi_v)$ for each nonarchimedean place v for which $\mathfrak{p}_v \cap \mathcal{O}_F$ divides \mathfrak{q}' . \square

4. COUNTING MONOMIAL REPRESENTATIONS

Let E be a quadratic extension of F . Via global class field theory, there is a quadratic Hecke character $\omega_{\text{quad}(E/F)}$ of $F^\times \backslash \mathbb{A}_F^\times$ associated to E/F ; its arithmetic conductor is $\mathfrak{d}_{E/F}$, the discriminant of E/F . Conversely, any quadratic Hecke character $\omega_{\text{quad}(\mathfrak{q}')}$ of arithmetic conductor \mathfrak{q}' corresponds to a quadratic extension E/F with discriminant $\mathfrak{d}_{E/F} = \mathfrak{q}'$.

Let χ be a Hecke character of $E^\times \backslash \mathbb{A}_E^\times$ of arithmetic conductor \mathfrak{F} such that χ does not factor through the global norm map $N_{\mathbb{A}_E/\mathbb{A}_F} : \mathbb{A}_E^\times \rightarrow \mathbb{A}_F^\times$. Then by the

work of Jacquet and Langlands [JL70, §12], there exists a cuspidal automorphic representation $\pi(\chi)$ of $\mathrm{GL}_2(\mathbb{A}_F)$ automorphically induced from χ . The representation $\pi(\chi)$ has central character $\omega_{\pi(\chi)} = \omega_{\mathrm{quad}(E/F)}\chi|_{\mathbb{A}_F^\times}$ and arithmetic conductor $\mathfrak{q} = N_{E/F}(\mathfrak{F})\mathfrak{d}_{E/F}$, where $N_{E/F}: E^\times \rightarrow F^\times$ denotes the norm map, and the integral ideal \mathfrak{F} of \mathcal{O}_E is the arithmetic conductor of χ . In particular, $\mathfrak{d}_{E/F}$ divides \mathfrak{q} .

This result is the most basic form of automorphic induction. We require a converse result. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, and suppose that there exists some nontrivial Hecke character ω of $F^\times \backslash \mathbb{A}_F^\times$ such that $\pi \otimes \omega \cong \pi$. Then π is said to be a monomial representation or of CM-type. The Hecke character ω must be quadratic, and so corresponds to a quadratic extension E of F via class field theory.

Theorem 4.1. *Let π be a monomial cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of arithmetic conductor \mathfrak{q} and central character ω_π , so that there exists a quadratic extension E of F such that $\pi \otimes \omega_{\mathrm{quad}(E/F)} \cong \pi$. Then π is automorphically induced from a Hecke character χ of $E^\times \backslash \mathbb{A}_E^\times$, so that $\pi \cong \pi(\chi)$. This character satisfies $\chi|_{\mathbb{A}_F^\times} = \omega_{\mathrm{quad}(E/F)}\omega_\pi$, has arithmetic conductor \mathfrak{F} satisfying $N_{E/F}(\mathfrak{F}) = \mathfrak{q}\mathfrak{d}_{E/F}^{-1}$, and does not factor through the global norm map $N_{\mathbb{A}_E/\mathbb{A}_F}$.*

Proof. This is the converse to automorphic induction for cuspidal automorphic representations of $\mathrm{GL}_1(\mathbb{A}_E)$ (that is, for Hecke characters), and is a result of Labesse and Langlands [LL79, Proposition 6.5]. \square

Given a monomial cuspidal automorphic representation π with trivial central character, let $\chi = \prod_w \chi_w$ be the Hecke character of $E^\times \backslash \mathbb{A}_E^\times$ from which π is automorphically induced, where the product is over the places w of E ; we will write $S_f(E), S_\infty(E), S_\mathbb{R}(E), S_\mathbb{C}(E)$ to denote the nonarchimedean, archimedean, real, and complex places of E respectively. Note that as $\chi|_{\mathbb{A}_F^\times} = \omega_{\mathrm{quad}(E/F)}$, with the latter having arithmetic conductor $\mathfrak{d}_{E/F}$, it follows that $\mathfrak{d}_{E/F}$ divides $N_{E/F}(\mathfrak{F})$, and hence $\mathfrak{q} = N_{E/F}(\mathfrak{F})\mathfrak{d}_{E/F}$ must be divisible by $\mathfrak{d}_{E/F}^2$. By the work of Jacquet and Langlands, we may describe the relation between π and χ at the archimedean places of F as follows.

Now if $v \in S_\mathbb{R}(F)$ splits in E , we have that $\pi_v \cong \chi_{w_1} \boxplus \chi_{w_2}$ for the two places $w_1, w_2 \in S_\mathbb{R}(E)$ lying over v , with $\chi_{w_1}(x) = \mathrm{sgn}_{w_1}(x)^{m_{w_1}}|x|_{w_1}^{s_{w_1}}$, $\chi_{w_2}(x) = \mathrm{sgn}_{w_2}(x)^{m_{w_2}}|x|_{w_2}^{s_{w_2}}$ for some $m_{w_1}, m_{w_2} \in \mathbb{Z}/2\mathbb{Z}$ and $s_{w_1}, s_{w_2} \in i\mathbb{R}$; as π_v has trivial central character, we have that $m_{w_2} = m_{w_1}$ and $s_{w_2} = -s_{w_1}$, so that the weight and spectral parameter of π_v are $k_v = 0$ and $s_v = s_{w_1} \in i\mathbb{R}$.

If $v \in S_\mathbb{R}(F)$ ramifies in E , so that there exists one complex place $w \in S_\mathbb{C}(E)$ lying over v , we have that $\pi_v \cong \mathcal{D}(\omega_{v,1}, \omega_{v,2})$ with $\omega_{v,1}, \omega_{v,2}$ characters of F_v^\times such that $\mathcal{D}(\omega_{v,1}, \omega_{v,2})$ corresponds to the character $\chi_w(z) = e^{im_w \arg_w(z)}|z|_w^{s_w}$ of E_w^\times , with $s_w = 0$, and $k_v = |m_w| + 1 \in 2\mathbb{N}$ and $s_v = m_w/2 \in 1/2 + \mathbb{Z}$ the weight and spectral parameter of π_v respectively.

Finally, every complex place v of F splits in E , so that for the two places $w_1, w_2 \in S_\mathbb{C}(E)$ lying over v , we have that $\pi_v \cong \chi_{w_1} \boxplus \chi_{w_2}$ with $\chi_{w_1}(z) = e^{im_{w_1} \arg_{w_1}(z)}|z|_{w_1}^{s_{w_1}}$, $\chi_{w_2}(z) = e^{im_{w_2} \arg_{w_2}(z)}|z|_{w_2}^{s_{w_2}}$ for some $m_{w_1}, m_{w_2} \in \mathbb{Z}$ and $s_{w_1}, s_{w_2} \in i\mathbb{R}$. We have that $m_{w_2} = -m_{w_1}$ and $s_{w_2} = -s_{w_1}$, so that the weight and spectral parameter of π_v are $k_v = 2|m_{w_1}| \in 2\mathbb{N}$ and $s_v = s_{w_1} \in i\mathbb{R}$.

So given a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, we write $S_\mathbb{R}^{\mathrm{ps}}(F)$ and $S_\mathbb{R}^{\mathrm{ds}}(F)$ for the set of real places v of F for which π_v is a principal series representation and a discrete series representation respectively, and we write $S_\infty^{\mathrm{ps}}(F) = S_\mathbb{R}^{\mathrm{ps}}(F) \cup S_\mathbb{C}(F)$ for the set of all archimedean places v for which π_v is a principal series representation. Now fix a nonempty subset $S_\infty^{\mathrm{ps}}(F) = S_\mathbb{R}^{\mathrm{ps}}(F) \cup S_\mathbb{C}(F)$ of the

set of archimedean places $S_\infty(F)$ of F , and let $\omega_{\text{quad}(\mathfrak{q}')}$ be a quadratic Hecke character of $F^\times \backslash \mathbb{A}_F^\times$ of arithmetic conductor \mathfrak{q}' . Then π can only be monomial with respect to $\omega_{\text{quad}(\mathfrak{q}')}$ if the quadratic extension E of F of discriminant $\mathfrak{d}_{E/F} = \mathfrak{q}'$ associated to $\omega_{\text{quad}(\mathfrak{q}')}$ via class field theory is such that an archimedean place v of F splits in E if and only if $v \in S_\infty^{\text{ps}}(F)$.

We wish to count cuspidal automorphic representations of $\text{GL}_2(\mathbb{A}_F)$ that are monomial with respect to $\omega_{\text{quad}(\mathfrak{q}')}$, have trivial central character, arithmetic conductor \mathfrak{q} , even archimedean weights \mathbf{k} , and principal series representations at a nonempty subset $S_\infty^{\text{ps}}(F)$ of the archimedean places of F . We let

$$\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}})_{\text{mon}(\omega_{\text{quad}(\mathfrak{q}')})}$$

denote the set of all such isomorphism classes of cuspidal automorphic representations.

Proposition 4.2. *Let $S_\infty^{\text{ps}} \subset S_\infty$ be nonempty and let \mathbf{k} be fixed even archimedean weights. Let \mathfrak{q} and \mathfrak{q}' be integral ideals of \mathcal{O}_F such that \mathfrak{q}' is squarefree, \mathfrak{q}'^2 divides \mathfrak{q} , and every prime ideal \mathfrak{p} dividing \mathfrak{q}' is nonradial, and let $\omega_{\text{quad}(\mathfrak{q}')}$ be a quadratic Hecke character of arithmetic conductor \mathfrak{q}' . We have that*

$$(5) \quad \left\{ \pi \in \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}})_{\text{mon}(\omega_{\text{quad}(\mathfrak{q}')})} : |\mathbf{s}_{\text{ps}}(\pi)| \leq \mathbf{T}_{\text{ps}} \right\} \ll_{F, \mathfrak{q}, \mathbf{k}} \prod_{v \in S_\infty^{\text{ps}}(F)} T_v.$$

Here we write $|\mathbf{s}_{\text{ps}}(\pi)| \leq \mathbf{T}_{\text{ps}}$ to mean that π has archimedean principal series spectral parameters $s_v(\pi_v)$ satisfying $|s_v(\pi_v)| \leq T_v$ for all $v \in S_\infty^{\text{ps}}(F)$, and the implicit constant in the error term depends on F , \mathfrak{q} , and \mathbf{k} .

Proof. Let E be the quadratic extension of F associated to $\omega_{\text{quad}(\mathfrak{q}')}$ via global class field theory. By Theorem 4.1, it suffices to count Hecke characters χ of E satisfying $\chi|_{\mathbb{A}_F^\times} = \omega_{\text{quad}(\mathfrak{q}')}$, having arithmetic conductor \mathfrak{F} such that $N_{E/F}(\mathfrak{F}) = \mathfrak{q}\mathfrak{d}_{E/F}^{-1}$, whose associated archimedean weights \mathbf{k} are fixed and archimedean principal series spectral parameters \mathbf{s}_{ps} are less than \mathbf{T}_{ps} in absolute value. Here if $v \in S_\infty^{\text{ps}}(F)$ with $w_1, w_2 \mid v$, then $k_v = |m_{w_1} - m_{w_2}| = 0$ and $s_v = (s_{w_1} - s_{w_2})/2 = s_{w_1}$, while if $v \in S_\infty^{\text{ds}}(F)$ with $w \mid v$, then $k_v = |m_w| + 1$, $s_v = m_w/2$, and $s_w = 0$, and finally $k_v = |m_{w_1} - m_{w_2}| = 2|m_{w_1}| \in 2\mathbb{N}$ and $s_v = (s_{w_1} - s_{w_2})/2 = s_{w_1} \in i\mathbb{R}$ if $v \in S_\infty^{\text{c}}(F)$ with $w_1, w_2 \mid v$.

Let $\chi = \prod_w \chi_w$ be such a character. The fact that χ is trivial on E^\times implies that for each unit ϵ of \mathcal{O}_E ,

$$1 = \prod_{w \in S_\infty(E)} \text{sgn}_w(\epsilon_w)^{m_w} |\epsilon_w|_w^{s_w} \prod_{w \in S_\infty(E)} e^{im_w \arg_w(\epsilon_w)} |\epsilon_w|_w^{s_w} \prod_{\substack{w \in S_f(E) \\ \mathfrak{P}_w \cap \mathcal{O}_E \mid \mathfrak{F}}} \chi_w(\epsilon_w),$$

where we write ϵ_w to denote the image of ϵ in the local field E_w , and for $w \in S_f(E)$, \mathfrak{P}_w is the maximal ideal of \mathcal{O}_w . It follows that

$$(6) \quad \sum_{\substack{v \in S_\infty^{\text{ps}}(F) \\ w_1, w_2 \mid v}} s_v \log \frac{|\epsilon_w|_{w_1}}{|\epsilon_w|_{w_2}} + \sum_{w \in S_\infty(E)} m_w \log \text{sgn}_w(\epsilon_w) \\ + i \sum_{w \in S_\infty(E)} m_w \arg_w(\epsilon_w) + \sum_{\substack{w \in S_f(E) \\ \mathfrak{P}_w \cap \mathcal{O}_E \mid \mathfrak{F}}} \log \chi_w(\epsilon_w) \in 2\pi i\mathbb{Z}$$

for a fixed branch of the logarithm that ensures this is well-defined. Each local character χ_w with $w \in S_f(E)$ and $\mathfrak{P}_w \cap \mathcal{O}_E \mid \mathfrak{F}$ descends to a primitive character of the finite abelian group $\mathcal{O}_w / (1 + \mathfrak{P}_w^{c(\chi_w)})$ when restricted to \mathcal{O}_w^\times ; as there are only finitely many such characters, there are only finitely many possible choices of

characters χ_w of E_w^\times up to multiplication by an unramified character. So we first fix a set of characters χ_w at each nonarchimedean place w of E for which $\mathfrak{P}_w \cap \mathcal{O}_E \mid \mathfrak{F}$.

We will show in Lemma 4.3 that

$$\left\{ \left(\log \frac{|\epsilon_w|_{w_1}}{|\epsilon_w|_{w_2}} \right)_{w_1, w_2 | v \in S_\infty^{\text{ps}}(F)} \in \mathbb{R}^{\#S_\infty^{\text{ps}}(F)} : \epsilon \in \mathcal{O}_E^\times \right\}$$

is a lattice in $\mathbb{R}^{\#S_\infty^{\text{ps}}(F)}$, and so

$$\left\{ \mathbf{s}_{\text{ps}} \in (i\mathbb{R})^{\#S_\infty^{\text{ps}}(F)} : (6) \text{ holds for all } \epsilon \in \mathcal{O}_E^\times \right\}$$

is a dual lattice in $(i\mathbb{R})^{\#S_\infty^{\text{ps}}(F)}$. The result then follows by standard bounds for the number of points in a lattice lying in a rectangle, and summing over the finitely many possible combinations of local characters χ_w for each $w \in S_f(E)$ for which $\mathfrak{P}_w \cap \mathcal{O}_E \mid \mathfrak{F}$. \square

Lemma 4.3. *Let E/F be a quadratic extension that is ramified at a nondyadic nonarchimedean place and is such that the set of archimedean places $S_\infty^{\text{ps}}(F)$ of F that split in E is nonempty, and fix an ordering w_1, w_2 of the places in E lying over a place $v \in S_\infty^{\text{ps}}(F)$. Consider the map $\Phi: \mathcal{O}_E^\times \rightarrow \mathbb{R}^{\#S_\infty^{\text{ps}}(F)}$ given by*

$$\Phi(\epsilon) := \left(\log \frac{|\epsilon_{w_1}|_{w_1}}{|\epsilon_{w_2}|_{w_2}} \right)_{w_1, w_2 | v \in S_\infty^{\text{ps}}(F)}.$$

Then Φ is a homomorphism with kernel $\mu(\mathcal{O}_E)\mathcal{O}_F^\times$, with $\mu(\mathcal{O}_E)$ the set of roots of unity lying in E , and \mathcal{O}_F^\times viewed as a subring of \mathcal{O}_E^\times . In particular, the image of Φ is a lattice in $\mathbb{R}^{\#S_\infty^{\text{ps}}(F)}$.

Proof. It is clear that Φ is a homomorphism and that $\mu(\mathcal{O}_E)\mathcal{O}_F^\times$ lies in the kernel of Φ . To prove the converse, we let σ be the nontrivial automorphism of E/F in $\text{Gal}(E/F)$. This acts transitively on the set of extensions E_w lying over F_v for any place v of F , so that for any $x \in E$, if a place $v \in S_\infty(F)$ splits in E with w_1, w_2 lying over v , then $\sigma(x_{w_2}) = x_{w_1}$, while if v ramifies in E with w lying over v , then $\sigma(x_w) = x_w$. It follows that $\epsilon \in \ker \Phi$ if and only if

$$\mu := \epsilon \sigma(\epsilon)^{-1}$$

satisfies $|\mu_w|_w = 1$ for every archimedean place w of E . As $\epsilon \in \mathcal{O}_E^\times$ implies that $\sigma(\epsilon) \in \mathcal{O}_E^\times$, so that $\mu \in \mathcal{O}_E^\times$, it follows that this is only possible should μ be a root of unity in E .

Let $N = mn$ be the order of the root of unity μ , with m an odd positive integer and n a power of 2. By Bézout's identity, there exist integers a, b such that $am + bn = 1$. We may then factor μ as $\mu = \mu^{am} \mu^{bn}$, with μ^{am}, μ^{bn} roots of unity in $\mu(\mathcal{O}_E)$ such that μ^{am} is of order n and μ^{bn} is of order $m = 2k + 1$.

We define the unit $\epsilon' := \epsilon \mu^{-(k+1)bn}$ of \mathcal{O}_E ; it satisfies

$$\sigma(\epsilon') = \sigma(\epsilon) \sigma(\mu)^{-(k+1)bn} = \epsilon \mu^{-1} \mu^{(k+1)bn} = \epsilon' \mu^{-am}$$

by the fact that $\sigma(\mu) = \mu^{-1}$ and that $\mu^{2(k+1)bn} = \mu^{bn}$. It follows that $\sigma(\epsilon'^n) = \epsilon'^n$, and hence ϵ'^n lies in F . We claim that $\epsilon' \in F$, in which case it must be a unit of \mathcal{O}_F , and we deduce that $\epsilon \in \mu(\mathcal{O}_F)\mathcal{O}_F^\times$, as desired.

Indeed, suppose that $n > 1$ is a power of 2 and that $\epsilon'^{n/2} \notin F$, and consider the extension $F(\epsilon'^{n/2})$ of F ; as $\epsilon'^n \in F$, this is a quadratic extension, and as $\epsilon'^{n/2} \in E$, this extension must be equal to E . On the other hand, this extension is generated by the square root of a unit ϵ'^n of \mathcal{O}_F , and hence is unramified at every nondyadic nonarchimedean place. As this contradicts our assumption on E , it follows that $\epsilon'^{n/2} \in F$. As n is a power of 2, we may iterate this method in order to conclude that $\epsilon' \in F$.

Finally, the fact that the image of Φ is a lattice follows from Dirichlet's unit theorem, which implies that $\mathcal{O}_E^\times / \ker \Phi$ is a finitely generated abelian group of rank

$$\#S_\infty(E) - 1 - \#S_\infty(F) + 1 = \#S_\infty^{\text{ps}}(F). \quad \square$$

5. WEYL LAWS FOR AUTOMORPHIC REPRESENTATIONS OF $\text{GL}_2(\mathbb{A}_F)$

Let v be a nonarchimedean place of an algebraic number field F , and let π_v, π'_v be generic irreducible admissible unitarisable representations of $\text{GL}_2(F_v)$. We say that π_v and π'_v are similar if they have the same central character and are both supercuspidal representations that are isomorphic up to twisting by the unramified character $|\cdot|_v^{\frac{\pi i}{\log q_v}}$, both special representations that are isomorphic up to twisting by the unramified character $|\cdot|_v^{\frac{\pi i}{\log q_v}}$, or they are both principal series representations

$$\pi_v \cong \beta_{v,1} | \cdot |_v^{s_1} \boxplus \beta_{v,2} | \cdot |_v^{s_2}, \quad \pi'_v \cong \beta'_{v,1} | \cdot |_v^{s'_1} \boxplus \beta'_{v,2} | \cdot |_v^{s'_2}$$

with $s_1 + s_2 - s'_1 - s'_2 \in \frac{2\pi i}{\log q_v} \mathbb{Z}$ and either $\beta'_{v,1} = \beta_{v,1}$ and $\beta'_{v,2} = \beta_{v,2}$, or $\beta'_{v,1} = \beta_{v,2}$ and $\beta'_{v,2} = \beta_{v,1}$. Likewise, for a real place v and generic irreducible admissible unitarisable $(\mathfrak{gl}_2(F_v), K_v)$ -modules π_v, π'_v , we say that π_v and π'_v are similar if they are both isomorphic discrete series representations or they are both principal series representations

$$\pi_v \cong \text{sgn}_v^{m_1} | \cdot |_v^{s_1} \boxplus \text{sgn}_v^{m_2} | \cdot |_v^{s_2}, \quad \pi'_v \cong \text{sgn}_v^{m'_1} | \cdot |_v^{s'_1} \boxplus \text{sgn}_v^{m'_2} | \cdot |_v^{s'_2}$$

with $s_1 + s_2 = s'_1 + s'_2$ and equal weights $k_v = |m_1 - m_2|$, $k'_v = |m'_1 - m'_2|$. Finally, two generic irreducible admissible unitarisable $(\mathfrak{gl}_2(F_v), K_v)$ -modules π_v, π'_v , with v a complex place of F , are said to be similar if they are both principal series representations

$$\pi_v \cong e^{im_1 \arg v} | \cdot |_v^{s_1} \boxplus e^{im_2 \arg v} | \cdot |_v^{s_2}, \quad \pi'_v \cong e^{im'_1 \arg v} | \cdot |_v^{s'_1} \boxplus e^{im'_2 \arg v} | \cdot |_v^{s'_2}$$

with $s_1 + s_2 = s'_1 + s'_2$ and equal weights $k_v = |m_1 - m_2|$, $k'_v = |m'_1 - m'_2|$.

In each case above, we let \mathfrak{X}_v denote the set of all such generic irreducible admissible unitarisable representations of $\text{GL}_2(F_v)$ or $(\mathfrak{gl}_2(F_v), K_v)$ -modules that are similar; we call \mathfrak{X}_v a local similarity class. A global similarity class $\mathfrak{X} := (\mathfrak{X}_v)$ of cuspidal automorphic representations of $\text{GL}_2(\mathbb{A}_F)$ is then defined to be the set of all isomorphism classes of cuspidal automorphic representations π of $\text{GL}_2(\mathbb{A}_F)$ for which $\pi_v \in \mathfrak{X}_v$ for each local component π_v of π . Note that two elements π, π' of a global similarity class \mathfrak{X} may well have different principal series spectral parameters s_v at each place v for which \mathfrak{X}_v is a similarity class of principal series representations, but that they share the same central character ω_π , arithmetic conductor \mathfrak{q} , and archimedean weights \mathbf{k} .

In [Pal12], Palm carefully chooses a test function for the adèlic Arthur–Selberg trace formula in order to produce a Weyl law for global similarity classes \mathfrak{X} of cuspidal automorphic representations; that is, he gives an asymptotic estimate for the number of elements of \mathfrak{X} having archimedean principal series spectral parameters s_{ps} bounded in absolute value by T_{ps} .

Proposition 5.1 (Weyl Law for $\text{GL}_2(\mathbb{A}_F)$ [Pal12, Theorem 3.2.1]). *Let \mathfrak{X} be a global similarity class of cuspidal automorphic representations of $\text{GL}_2(\mathbb{A}_F)$, and let S_∞^{ps} be a nonempty subset of the archimedean places of F such that \mathfrak{X}_v is a similarity class of principal series representations for each $v \in S_\infty^{\text{ps}}$. Suppose further that at each complex place v , the archimedean weight of \mathfrak{X}_v is $k_v = 0$. Then we have the*

Weyl law

$$\begin{aligned} & \# \{ \pi \in \mathfrak{X} : |\mathbf{s}_{\text{ps}}(\pi)| \leq \mathbf{T}_{\text{ps}} \} \\ &= C_{\mathfrak{X}} \prod_{v \in S_{\mathbb{R}}^{\text{ps}}} T_v^2 \prod_{v \in S_{\mathbb{C}}} T_v^3 \\ &+ O_F \left(C_{\mathfrak{X}} \sum_{v' \in S_{\infty}^{\text{ps}}} \frac{1}{T_{v'}} \prod_{v \in S_{\mathbb{R}}^{\text{ps}}} T_v^2 \prod_{v \in S_{\mathbb{C}}} T_v^3 + \sum_{v' \in S_{\mathbb{R}}^{\text{ps}}} \log T_{v'} \prod_{v \in S_{\infty}^{\text{ps}}} T_v \right). \end{aligned}$$

Here the implicit constant in the error term depends only on the algebraic number field F , and the constant $C_{\mathfrak{X}}$ is defined by

$$C_{\mathfrak{X}} := \frac{\text{vol}(Z(\mathbb{A}_F) \text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A}_F))}{(4\pi)^{\#S_{\mathbb{R}}}(8\pi^2)^{\#S_{\mathbb{C}}}} \prod_{v \notin S_{\infty}^{\text{ps}}} C_{\mathfrak{X}_v},$$

with each local constant $C_{\mathfrak{X}_v}$ defined such that if \mathfrak{X}_v is not a similarity class of principal series representations, then for any $\pi_v \in \mathfrak{X}_v$,

$$C_{\mathfrak{X}_v} := \begin{cases} q_v - 1 & \text{if } \pi_v \text{ is a special representation,} \\ \dim \rho_{\pi_v} & \text{if } \pi_v \text{ is a type I supercuspidal representation,} \\ \frac{q_v + 1}{2} \dim \rho_{\pi_v} & \text{if } \pi_v \text{ is a type II supercuspidal representation,} \\ k_v - 1 & \text{if } \pi_v \text{ is a discrete series representation of weight } k_v, \end{cases}$$

while if \mathfrak{X}_v is a similarity class of principal series representations, then for any $\pi_v \cong \omega_{v,1} \boxplus \omega_{v,2} \in \mathfrak{X}_v$,

$$C_{\mathfrak{X}_v} := \begin{cases} 2 \frac{q_v^{c(\omega_{v,1}\omega_{v,2}^{-1})} + q_v^{c(\omega_{v,1}\omega_{v,2}^{-1})-1}}{q_v^{\lfloor c(\omega_{v,1}\omega_{v,2}^{-1})/2 \rfloor} + 1} & \text{if } c(\omega_{v,1}\omega_{v,2}^{-1}) \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

We have corrected several typographical errors in [Pal12, Theorem 3.2.1] in the definition of $C_{\mathfrak{X}}$. By [Pal12, Proposition 7.7.1], [Pal12, Corollary 7.7.2], and [Pal12, Proposition 8.7.1], the denominator in the definition of $C_{\mathfrak{X}}$ should be $(4\pi)^{\#S_{\mathbb{R}}}(8\pi^2)^{\#S_{\mathbb{C}}}$, not $(4\pi)^{\#S_{\infty}}$. The definition of $C_{\mathfrak{X}_v}$ for \mathfrak{X}_v a similarity class of discrete series representations of weight k_v ought to be $k_v - 1$ as in [Pal12, Corollary 7.7.2], not $\frac{k_v-1}{2}$. Finally, for \mathfrak{X}_v a similarity class of ramified principal series representations, the definition for $C_{\mathfrak{X}_v}$ is incorrect in both [Pal12, Theorem 3.2.1] and its derivation in [Pal12, Proposition 9.7.1]; the latter is missing a factor of 2 that should be present due to the fact that $\omega_{v,1} \boxplus \omega_{v,2}$ and $\omega_{v,1} \cdot \left| \frac{\pi i}{\log q_v} \right| \boxplus \omega_{v,2} \cdot \left| \frac{\pi i}{\log q_v} \right|$ lie in the same similarity class. One can check that these are the correct definitions by comparing Palm's result to, say, the classical Weyl law for $\mathcal{M}^{\text{new}}(\Gamma_0(p^2))$ for p an odd prime as found in [Ris04], using the fact that

$$\text{vol}(Z(\mathbb{A}_{\mathbb{Q}}) \text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}_{\mathbb{Q}})) = \text{vol}(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}) = \frac{\pi}{3}.$$

We let

$$\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_{\infty}^{\text{ps}})$$

denote the set of isomorphism classes of cuspidal automorphic representations of arithmetic conductor \mathfrak{q} , trivial central character, archimedean weights \mathbf{k} , and having principal series representations at a fixed nonempty subset S_{∞}^{ps} of the archimedean places of F ; this is a union of similarity classes. For a quadratic Hecke character

$\omega_{\text{quad}(\mathfrak{q}')}$, we define

$$\begin{aligned} \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}})_{\text{nonmon}(\omega_{\text{quad}(\mathfrak{q}')})} \\ := \{ \pi \in \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}}) : \pi \otimes \omega_{\text{quad}(\mathfrak{q}')} \not\cong \pi, \\ \pi \otimes \omega_{\text{quad}(\mathfrak{q}')} \in \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}}) \} ; \end{aligned}$$

these are the isomorphism classes of nonmonomial cuspidal automorphic representations in $\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}})$ that are conductor-invariant under twisting by $\omega_{\text{quad}(\mathfrak{q}')}$. Using the Weyl law, we are able to count such cuspidal automorphic representations, and hence prove instances of spectral multiplicity for modular forms over arbitrary number fields.

Theorem 5.2. *Let $S_\infty^{\text{ps}} \subset S_\infty$ be nonempty and \mathbf{k} be fixed even archimedean weights with $k_v = 0$ for all $v \in S_\infty^{\text{ps}}$. Let \mathfrak{q} and \mathfrak{q}' be integral ideals of \mathcal{O}_F such that \mathfrak{q}' is squarefree and every prime ideal \mathfrak{p} dividing \mathfrak{q}' is nondividing. Let $\omega_{\text{quad}(\mathfrak{q}')}$ be a quadratic Hecke character of arithmetic conductor \mathfrak{q}' . Then if there exists a prime ideal \mathfrak{p} dividing \mathfrak{q}' such that \mathfrak{p}^2 does not divide \mathfrak{q} , we have that*

$$(7) \quad \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}})_{\text{nonmon}(\omega_{\text{quad}(\mathfrak{q}')})} = \emptyset,$$

while if \mathfrak{p}^2 divides \mathfrak{q} for every prime ideal \mathfrak{p} dividing \mathfrak{q}' , we have the Weyl law

$$(8) \quad \frac{\# \left\{ \pi \in \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}})_{\text{nonmon}(\omega_{\text{quad}(\mathfrak{q}')})} : |\mathbf{s}_{\text{ps}}(\pi)| \leq \mathbf{T}_{\text{ps}} \right\}}{\# \left\{ \pi \in \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}}) : |\mathbf{s}_{\text{ps}}(\pi)| \leq \mathbf{T}_{\text{ps}} \right\}} \\ = \prod_{\substack{v \in S_f \\ \mathfrak{p}_v \cap \mathcal{O}_F \mid \mathfrak{q}' \\ (\mathfrak{p}_v \cap \mathcal{O}_F)^2 \nmid \mathfrak{q}}} \left(1 - \frac{q_v}{q_v^2 - q_v - 1} \right) + o_{F, \mathfrak{q}, \mathbf{k}}(1).$$

Here the error term depends on F, \mathfrak{q} , and \mathbf{k} , and we have the convention that the empty product is equal to 1.

Moreover, the same holds if we replace $\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}})_{\text{nonmon}(\omega_{\text{quad}(\mathfrak{q}')})}$ by

$$(9) \quad \bigcap_{\substack{\mathfrak{q}^* \mid \mathfrak{q}' \\ \mathfrak{q}^* \neq \mathcal{O}_F}} \bigcap_{\substack{\omega_{\text{quad}(\mathfrak{q}^*)} \\ \text{cond}(\omega_{\text{quad}(\mathfrak{q}^*)}) = \mathfrak{q}^*}} \mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}})_{\text{nonmon}(\omega_{\text{quad}(\mathfrak{q}^*)})},$$

where $\text{cond}(\omega)$ denotes the arithmetic conductor of ω , and the intersection is over all integral ideals \mathfrak{q}^* of \mathcal{O}_F dividing \mathfrak{q}' other than \mathcal{O}_F and all quadratic characters $\omega_{\text{quad}(\mathfrak{q}^*)}$ of arithmetic conductor \mathfrak{q}^* .

The special case $F = \mathbb{Q}$ and $k_\infty = 0$ of Theorem 5.2 is Theorem 1.3 by the bijective correspondence between newforms and cuspidal automorphic representations. Note in this case that there is a unique primitive quadratic character modulo q^* for every positive odd integer q^* greater than 1.

Proof. It is clear from Theorem 3.2 that (7) holds when there exists a prime ideal \mathfrak{p} dividing \mathfrak{q}' such that \mathfrak{p}^2 does not divide \mathfrak{q} . If \mathfrak{p}^2 divides \mathfrak{q} for every prime ideal \mathfrak{p} dividing \mathfrak{q}' , then the monomial representations have density zero in $\mathfrak{X}(\Gamma_0(\mathfrak{q}), \mathbf{k}, S_\infty^{\text{ps}})$ by Proposition 4.2 and Proposition 5.1. So by Proposition 5.1 and Theorem 3.2, to show (8) we must merely determine the ratio of the quantities

$$\sum_{\substack{\mathfrak{X}_v \ni \pi_v \\ c(\pi_v)=2 \\ \omega_{\pi_v}=1}} C_{\mathfrak{X}_v} - \sum_{\substack{\mathfrak{X}_v \ni \pi_v \\ \pi_v \cong \beta_{v, \text{quad}} \boxplus \beta_{v, \text{quad}}}} C_{\mathfrak{X}_v} - \sum_{\substack{\mathfrak{X}_v \ni \pi_v \\ \pi_v \cong \beta_{v, \text{quad}} \text{St}_v}} C_{\mathfrak{X}_v},$$

and

$$\sum_{\substack{\mathfrak{X}_v \ni \pi_v \\ c(\pi_v)=2 \\ \omega_{\pi_v}=1}} C_{\mathfrak{X}_v}$$

for each place $v \in S_f$ with $\mathfrak{p}_v \cap \mathcal{O}_F \mid \mathfrak{q}'$ and $(\mathfrak{p}_v \cap \mathcal{O}_F)^2 \parallel \mathfrak{q}'$. The product of these ratios over all such places then yields (8). Finally, the fact that we can replace $\mathfrak{X}(\Gamma_0(q), \mathbf{k}, S_\infty^{\text{ps}})_{\omega_{\text{quad}}(q')}$ by (9) again follows from Theorem 3.2 and the fact that the monomial representations have density zero.

There is only one similarity class of special representations of conductor \mathfrak{p}_v^2 and trivial central character, namely $\mathfrak{X}_v \ni \pi_v$ with $\pi_v \cong \beta_{v,\text{quad}} \text{St}_v$; this has $C_{\mathfrak{X}_v} = q_v - 1$. Similarly, there is only one similarity class \mathfrak{X}_v of principal series representations with $\pi_v \cong \omega_{v,1} \boxplus \omega_{v,2} \in \mathfrak{X}_v$ satisfying $c(\omega_{v,1}\omega_{v,2}^{-1}) = 0$, namely $\mathfrak{X}_v \ni \pi_v$ with $\pi_v \cong \beta_{v,\text{quad}} \boxplus \beta_{v,\text{quad}}$; this has $C_{\mathfrak{X}_v} = 1$.

The remaining similarity classes of principal series representations are of the form $\mathfrak{X}_v \ni \pi_v \cong \omega_v \boxplus \omega_v^{-1}$ with $c(\omega_v) = c(\omega_v^2) = 1$. These are parametrised by the square of characters β_v of $\mathcal{O}_v^\times / (1 + \mathfrak{p}_v)$ with $\beta_v^2 \neq 1$; there are

$$\frac{\#\mathcal{O}_v^\times / (1 + \mathfrak{p}_v)}{2} - 1 = \frac{q_v - 3}{2}$$

distinct squares of these characters, with $C_{\mathfrak{X}_v} = q_v + 1$ in each case.

Finally, Knightly and Ragsdale [KR14] show that every supercuspidal representation π_v of conductor \mathfrak{p}_v^2 is of type I with $\dim \rho_{\pi_v} = q_v - 1$, and the number of isomorphism classes of such representations having trivial central character is $\frac{q_v - 1}{2}$. So for each such similarity class \mathfrak{X}_v , we have that $C_{\mathfrak{X}_v} = q_v - 1$.

Combining these calculations, we find that

$$\sum_{\substack{\mathfrak{X}_v \ni \pi_v \\ c(\pi_v)=2 \\ \omega_{\pi_v}=1}} C_{\mathfrak{X}_v} = q_v - 1 + 1 + \frac{q_v - 3}{2} (q_v + 1) + \frac{q_v - 1}{2} (q_v - 1) = q_v^2 - q_v - 1,$$

and that

$$\sum_{\substack{\mathfrak{X}_v \ni \pi_v \\ c(\pi_v)=2 \\ \omega_{\pi_v}=1}} C_{\mathfrak{X}_v} - \sum_{\substack{\mathfrak{X}_v \ni \pi_v \\ \pi_v \cong \beta_{v,\text{quad}} \boxplus \beta_{v,\text{quad}}}} C_{\mathfrak{X}_v} - \sum_{\substack{\mathfrak{X}_v \ni \pi_v \\ \pi_v \cong \beta_{v,\text{quad}} \text{St}_v}} C_{\mathfrak{X}_v} = q_v^2 - 2q_v - 1. \quad \square$$

6. CONSEQUENCES AND CONJECTURES

6.1. Spectral Multiplicity. While our results give lower bounds for spectral multiplicity, our methods say very little about corresponding upper bounds. In light of the numerical evidence of the simplicity of the discrete spectrum of the Laplacian on $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$, however, we believe that spectral multiplicity should only occur due to quadratic twisting, with the possible exception of the eigenvalue $1/4$.

Conjecture 6.1 (Spectral Multiplicity Conjecture for $\Gamma_0(q) \backslash \mathbb{H}$). *Let q be an odd positive integer, and let $s(q)$ denote the number of distinct odd primes p for which p^2 divides q . Then the multiplicity of an eigenvalue $\lambda > 1/4$ in the new part of the discrete spectrum of the Laplacian on $\Gamma_0(q) \backslash \mathbb{H}$ is bounded from above by $2^{s(q)}$, with this bound attained for a positive proportion of eigenvalues.*

When q is even, the situation is more complicated. Indeed, throughout this paper we have stipulated that all quadratic characters have odd conductor. Classifying the level of twists of newforms by quadratic characters of even conductor has been considered by Atkin and Lehner [AL70, Theorem 7] for holomorphic newforms, with the chief challenge being that there is no quadratic character modulo 2, one

quadratic character modulo 4, and two primitive quadratic characters modulo 8. It is likely that, at least for the case $F = \mathbb{Q}$, the methods of this paper could be used to extend results on spectral multiplicity to even values of q . We also remark that the discrete spectrum of the Laplacian on $\Gamma_0(q) \backslash \mathbb{H}$ with q even has been studied via the Selberg trace formula by Golovchanskii and Smotrov [GS12].

6.2. Distinguishing Newforms by Hecke Eigenvalues. A question closely related to the content of this article is the problem of distinguishing newforms by their Hecke eigenvalues. For holomorphic newforms of weight k , this issue has been studied by Chow and Ghitza [CG15]; for many small positive squarefree integers q and positive even integers k , they numerically calculate the least integer $n_0(q, k)$ such that if f and g are holomorphic newforms of weight k , level q , and principal character with equal Hecke eigenvalues $\lambda_f(p) = \lambda_g(p)$ for all $p \leq n_0(q, k)$, then $f = g$. Based on their calculations, they give the following conjecture.

Conjecture 6.2 (Stability Conjecture for Holomorphic Newforms [CG15, Conjecture 4.1]). *Let q be a positive squarefree integer and let k be a positive even integer. Then there exists a positive integer k_0 such that for all $k \geq k_0$, $n_0(q, k)$ is equal to the smallest prime not dividing q .*

When q is not squarefree, their numerical evidence suggests that this conjecture must be altered; in particular, for $q = 49$ it seems that $n_0(q, k) = 3$. This is due to the presence of distinct newforms f, g for which $f \otimes \varepsilon_{\text{quad}(7)} = g$; as $\varepsilon_{\text{quad}(7)}(2) = 1$, these newforms satisfy $\lambda_f(2) = \lambda_g(2)$.

We may well ask the same problem of distinguishing Maaß newforms by their Hecke eigenvalues. Here we must make the restriction that a newform is nondegenerate in the sense that it is nonmonomial and does not have Laplacian eigenvalue $1/4$. Then we let $n_0(q, T)$ denote the least integer such that if φ and ψ are nonmonomial Maaß newforms of weight 0, level q , and principal character with Laplacian eigenvalues $1/4 < \lambda_\varphi, \lambda_\psi \leq T$ and identical Hecke eigenvalues $\lambda_\varphi(p) = \lambda_\psi(p)$ for all $p \leq n_0(q, T)$, then $\varphi = \psi$. Theorem 1.3 shows that if q is not squarefree, for each odd squarefree divisor q' of q there exists a positive proportion of Maaß newforms φ, ψ of weight 0, level q , and principal character are not equal yet satisfy $\varphi \otimes \varepsilon_{\text{quad}(q')} = \psi$, where $\varepsilon_{\text{quad}(q')}$ is the unique primitive quadratic character modulo q' . Consequently, these newforms satisfy $\lambda_\varphi(p) = \lambda_\psi(p)$ for every prime p for which $\varepsilon_{\text{quad}(q')}(p) = 1$. This leads us to suggest the following conjecture.

Conjecture 6.3 (Stability Conjecture for Maaß Newforms). *Let q be a positive odd squarefree integer. Then there exists a positive real T_0 such that for all $T \geq T_0$, $n_0(q, T)$ is equal to the smallest prime not dividing q if q is squarefree, while if q is non-squarefree then*

$$n_0(q, T) = \max_{\substack{q^* | q' \\ q^* > 1}} \left\{ \min \{ p : \varepsilon_{\text{quad}(q^*)}(p) = -1, p \nmid q' \} \right\},$$

where q' is the largest squarefree integer dividing q such that every prime p dividing q' is such that p^2 divides q , and $\varepsilon_{\text{quad}(q^*)}$ denotes the unique primitive quadratic character modulo q^* .

We would expect this stabilising value of $n_0(q, T)$ to also be the stabilising value of $n_0(q, k)$ for holomorphic newforms, again under the proviso that such newforms be nonmonomial. Indeed, let $\mathcal{S}_k^{\text{new}}(\Gamma_0(q))$ denote the space of holomorphic newforms of even weight $k \geq 2$, level $q \geq 1$, and principal character, and let $\mathcal{S}_k^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q')})}$ denote the subspace of $\mathcal{S}_k^{\text{new}}(\Gamma_0(q))$ spanned by newforms whose twist by a quadratic character $\varepsilon_{\text{quad}(q')}$ is a different holomorphic

newform of the same weight, level, and character. The methods developed in this paper can be used to show the following.

Theorem 6.4. *Let q and q' be positive integers with q' odd and squarefree. Let $\varepsilon_{\text{quad}(q')}$ denote the unique primitive quadratic character modulo q' . Then if there exists a prime p dividing q' such that p^2 does not divide q ,*

$$\mathcal{S}_k^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q')})} = \{0\},$$

whereas if p^2 divides q for every prime p dividing q' , we have that

$$\frac{\dim \mathcal{S}_k^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q')})}}{\dim \mathcal{S}_k^{\text{new}}(\Gamma_0(q))} = \prod_{\substack{p|q' \\ p^2 \nmid q}} \left(1 - \frac{p}{p^2 - p - 1}\right) + o_q(1)$$

as k tends to infinity over the even integers, where the error term depends only on q .

Moreover, the same holds if we replace $\mathcal{S}_k^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q')})}$ by

$$\bigcap_{\substack{q^*|q' \\ q^* > 1}} \mathcal{S}_k^{\text{new}}(\Gamma_0(q))_{\text{nonmon}(\varepsilon_{\text{quad}(q^*)})}.$$

6.3. Twists of Maaß Forms on $\Gamma_1(q)\backslash\mathbb{H}$. In this article, we classified nonmonomial Maaß newforms of principal character that are level- and character-invariant under twisting by a Dirichlet character, with such characters necessarily being quadratic. A related question is to relax the character invariance and classify the larger family of Maaß newforms, possibly having nonprincipal character, that are level-invariant under twisting by some Dirichlet character. More precisely, we consider the space

$$\mathcal{M}^{\text{new}}(\Gamma_1(q)) := \bigoplus_{\varepsilon \pmod{q}} \mathcal{M}^{\text{new}}(q, \varepsilon)$$

of Maaß newforms on $\Gamma_1(q)\backslash\mathbb{H}$ and ask when two distinct newforms φ, ψ in $\mathcal{M}^{\text{new}}(\Gamma_1(q))$ are related by $\varphi \otimes \varepsilon = \psi$ for some Dirichlet character ε .

Classifying these newforms would lead to a better understanding of the new part of the discrete spectrum of the Laplacian on $\Gamma_1(q)\backslash\mathbb{H}$, which of course contains the new part of the discrete spectrum of the Laplacian on $\Gamma_0(q)\backslash\mathbb{H}$. The chief difference in this setting is that spectral multiplicity occurs already for squarefree values of q , as was first observed by Booker and Strömbergsson [BS07, Section 3.4]. Indeed, if q is odd and squarefree, and $\mathcal{M}^{\text{new}}(q, \varepsilon)$ denotes the space of Maaß newforms of weight zero, level q , and character ε for some Dirichlet character ε modulo q satisfying $\varepsilon^2 \neq \varepsilon_{0(q)}$, then

$$\mathcal{M}^{\text{new}}(q, \varepsilon) \otimes \bar{\varepsilon} = \mathcal{M}^{\text{new}}(q, \bar{\varepsilon}).$$

Another notable difference to $\Gamma_0(q)\backslash\mathbb{H}$ is that if $q = p^m$ is a power of an odd prime, then the largest possible multiplicity of an eigenvalue can be shown to grow as the power m grows.

Proposition 6.5. *Let $q = p^m$ be a power of an odd prime p with $m \geq 4$. Then the new part of the discrete spectrum of the Laplacian on $\Gamma_1(q)\backslash\mathbb{H}$ contains eigenvalues of multiplicity at least $p^{\lfloor m/2 \rfloor - 2}(p - 1)^2$.*

Sketch of Proof. Take a newform for which the local component π_p at p of the associated cuspidal automorphic representation π is a principal series representation

$$\pi_p \cong \beta_{p,1}|\cdot|_p^{s_1} \boxplus \beta_{p,1}|\cdot|_p^{s_2},$$

where $\beta_{p,1}, \beta_{p,2}$ are characters of \mathbb{Z}_p^\times of conductor exponents $c(\beta_{p,1}) = \lceil m/2 \rceil$, $c(\beta_{p,2}) = \lfloor m/2 \rfloor$, and we choose $\beta_{p,2} = \beta_{p,1}$ if m is even; the conductor exponent of π_p is

$$c(\pi_p) = c(\beta_{p,1}) + c(\beta_{p,2}) = \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor = m.$$

For each character β'_p of \mathbb{Z}_p^\times of conductor exponent at most $\lfloor m/2 \rfloor$ satisfying $c(\beta_{p,2}\beta'_p) = \lfloor m/2 \rfloor$, the twist $\pi_p \otimes \beta'_p$ is a principal series representation

$$\beta_{p,1}\beta'_p| \cdot |_p^{s_1} \boxplus \beta_{p,2}\beta'_p| \cdot |_p^{s_2}$$

of conductor exponent

$$c(\beta_{p,1}\beta'_p) + c(\beta_{p,2}\beta'_p) = \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor = m,$$

as $c(\beta'_p) \leq \lfloor m/2 \rfloor$ implies that $c(\beta_{p,1}\beta'_p) = c(\beta_{p,1})$. It is clear that no two such twists by different characters are isomorphic. The number of possible twists is the number of characters $\beta_{p,2}\beta'_p$ of \mathbb{Z}_p^\times of conductor exponent $\lfloor m/2 \rfloor$, which is the number of primitive characters modulo $p^{\lfloor m/2 \rfloor}$; this is $p^{\lfloor m/2 \rfloor - 2}(p-1)^2$ if $m \geq 4$. \square

We leave unaddressed the complete classification of local representations that are conductor-invariant under twists, though the methods we developed in this article would certainly be capable of dealing with this problem. The main new issue would be classifying supercuspidal representations that are conductor-invariant under twisting by a character that is not necessarily quadratic; this would involve significantly more work than the simpler case in Lemma 2.5.

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